Robust Comparative Statics in Contests

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April 11, 2018

Abstract

We drive several robust comparative statics results in a contest under minimal restrictions on the primitives. Some of our findings generalize existing results, while others clarify the relevance of structure commonly imposed in the literature. Contrasting prior results, we show, via an example, that equilibrium payoffs may be (strictly) decreasing in the value of the prize. We also obtain a condition under which equilibrium aggregate activity decreases in the number of players. Finally, we shed light on equilibrium existence and uniqueness. Differentiating this study from past work is our reliance on lattice-theoretic techniques, which allows for a more general approach.

Keywords: Contests, Supermodularity, Entry, Comparative Statics
JEL Classifications: C61, C72, D72

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1 Introduction

Contests are games in which players exert effort to increase their chances of winning a prize.[1] In this paper, we study the properties of a symmetric contest under the logit contest success function (CSF). We explore how equilibrium behavior depends on the parameters of the model, and examine existence/uniqueness of equilibrium. Differentiating our analysis from previous work, we invoke tools from lattice theory, which allows a more general approach. To the best of our knowledge, this is the first study to apply these tools in contests.

The logit CSF generalizes the model popularized by Tullock (1980), and is frequently studied in the literature.[2] Prior work in this model relies on the Implicit Function Theorem to conduct comparative statics. This approach requires economically-relevant assumptions on the returns to scale of the contest technology, which may or may not be justified, or necessary for the results in question. This creates ambiguity in identifying the underlying economic mechanisms at play; our approach helps clarify this ambiguity.

As noted by Acemoglu and Jensen (2013), contests *cannot* be supermodular or submodular games. Nevertheless, by restricting attention to subsets of the strategy space, we are able to apply the tools from supermodular games to establish a strong regularity property on the shape of the best response correspondence. This property allows us to derive several comparative statics, which hold generically so long as a symmetric pure-strategy equilibrium exists. Our approach suggests a means for applying lattice-theoretic techniques in contests, which should help guide future work develop more robust conclusions.

The regularity property we establish on the best-response extends a finding from Dixit (1987), while our comparative statics extend/clarify results in Nti (1997), and complement Acemoglu and Jensen (2013) and Jensen (2016). Dixit and Nti impose decreasing returns to scale, and invoke the Implicit Function Theorem. Contrasting Nti's results, when the contest technology exhibits

[1]See Corchón (2007) and Konrad (2009) for overviews.
[2]See Skaperdas (1996) for an overview.

increasing returns to scale, we show that equilibrium payoffs may be strictly decreasing in the value of the prize, and that equilibrium aggregate activity may be decreasing in the number of players. Acemoglu and Jensen study a more general version of the logit CSF, and allow asymmetries between players. Jensen studies an asymmetric contest under the logit CSF. Both studies utilize tools from aggregative games, and rely on a "local solvability condition". In the special case of a symmetric contest with the logit CSF, this condition corresponds to our definition of decreasing returns to scale.

We also provide conditions that ensure existence of a pure-strategy equilibrium. For these results, we exploit an equivalence with the Cournot oligopoly model, and apply findings from Amir (1996) and Amir and Lambson (2000; henceforth AL). These results complement the analyses of Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), Yamakazi (2008), Acemoglu and Jensen (2013) and Jensen (2016). These studies consider asymmetric contests, but impose stronger assumptions on the returns-to-scale of the contest technology.[3]

The lattice-theoretic techniques we apply were developed by Topkis (1978), and introduced in economics by works such as Vives (1990), Milgrom and Shannon (1994) and Milgrom and Roberts (1994). For surveys see Vives (2005), Amir (2005) or Topkis (1998).

2 Model

We consider a contest in which n symmetric players compete for a single prize of common value, V. Each player, $i \in \{1, ..., n\}$, chooses an effort, $e_i \in \mathbb{R}_+$, according to the cost function, $C : \mathbb{R}_+ \to \mathbb{R}_+$. If player i chooses effort e_i, and the other $n-1$ players choose efforts according to $\mathbf{e}_{-i} \in \mathbb{R}_+^{n-1}$, the probability that i wins is given by the logit CSF:

[3]Cornes and Hartley allow for increasing returns to scale in the special case of the Tullock CSF.

$$P(e_i, \mathbf{e}_{-i}) = \frac{\phi(e_i)}{R + \sum_{j=1}^{n} \phi(e_j)}.$$

We refer to $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ as the production function, and we call $\phi(e_i)$ player i's "force". In settings such as patent races or inducement contests it may happen that no player wins; the "discount rate", $R \geq 0$, captures this possibility (see, e.g. Loury, 1979). We assume $R > 0$, but we discuss the case where $R = 0$ in Section 3.3. When $\phi(x) = x^r$, and $R = 0$, the logit CSF corresponds to the popular Tullock CSF. The expected payoff to player i is,

$$\hat{\pi}_i(e_i, \mathbf{e}_{-i}) = \frac{\phi(e_i)}{R + \sum_{j=1}^{n} \phi(e_j)} V - C(e_i).$$

Player i chooses e_i to maximize π_i, taking $\mathbf{e}_{-i}$ as given. We make the following assumptions:

Assumption 1.

(i) ϕ is continuous and strictly increasing.

(ii) C is lower semi continuous and increasing.

(iii) For all $\mathbf{e}_{-i} \in \mathbb{R}_+^{n-1}$, $\lim_{e \to \infty} \hat{\pi}(e, \mathbf{e}_{-i}) < 0$

Assumptions 1(i)-(ii) imply that greater effort strictly increases a player's likelihood of winning, but at a greater cost to the player. We do not require continuity of C, which has relevant economic content, as lower semi continuity allows for (avoidable) fixed costs. We also do not require $\phi(0) = 0$ or $C(0) = 0$, which allows for the possibility of past sunk investments in effort. Assumption 1(iii) implies that we may, without further loss of generality, restrict attention to effort choices $e \in [0, \overline{e}]$ for some arbitrarily large $\overline{e} > 0$.

A Recasting

Let $x \equiv \phi(e)$, $y \equiv \sum_{j \neq i} \phi(e_j)$, and $z \equiv x + y$ denote, respectively, the force allocated to the contest by some player i, the total force of all players other than

i, and the total force. Let $\underline{x} = \phi(0)$, $\overline{x} = \phi(\overline{e})$, $\underline{y} = (n-1)\underline{x}$, and $\overline{y} = (n-1)\overline{x}$. We can then think of the players as choosing forces directly. Since ϕ is strictly increasing and continuous, the inverse function, $\phi^{-1} : [\underline{x}, \overline{x}] \to [\underline{e}, \overline{e}]$, is well-defined, strictly increasing, and continuous. We let $\kappa = C \circ \phi^{-1}$, and re-write player i's payoff as follows:

$$\pi(x, y) = \frac{x}{R + x + y} V - \kappa(x). \tag{1}$$

Since the game is symmetric, we avoid the use of subscripts for ease of notation. For any $y \geq 0$ we let

$$X^*(y) = \arg\max \big\{ \pi(x, y) | x \in [\underline{x}, \overline{x}] \big\},$$

and let $r : \mathbb{R}_+ \to [\underline{x}, \overline{x}]$ denote an arbitrary single-valued selection from X^*. In our definition of X^* we allow y to take on any positive value, including those outside the feasible range of $[\underline{y}, \overline{y}]$. We can thus think of X^* as an extension of the best-response correspondence; but in a slight abuse of terminology, we refer to it simply as the best-response. Our assumptions on ϕ and C ensure that, for any fixed $y \geq 0$, $\pi(\cdot, y)$ is upper semi continuous; since the choice set is compact, $X^*(y)$ is non empty (and possibly set-valued).

3 Results

We divide our results into two sets. For our first set of results – in Section 3.1 – we impose no additional structure on the model. Although the existence of a pure-strategy equilibrium cannot be guaranteed, our results apply immediately in any setting where a symmetric pure-strategy equilibrium exists. For our second set of results – in Section 3.2 – we provide conditions that ensure existence of a pure-strategy equilibrium, and study how aggregate behavior depends on the number of players. We discuss the case where $R = 0$ in Section 3.3. All proofs are contained in the Appendix.

We focus on pure-strategy Nash equilibria; for brevity we simply write "equilibrium". We refer to ϕ and C jointly as the contest technology. We say that the technology exhibits decreasing [increasing] returns to scale if κ is convex [concave]. Throughout this paper, when we write "increasing" or "decreasing" we mean in the weak sense; otherwise we shall write "strictly increasing" or "strictly decreasing".

Finally, borrowing terminology from Jensen, we say i is an "absolute favorite to lose [win]" if $\frac{x}{R+x+y} < [>]\frac{1}{2}$; equivalently, $x < [>] y + R$. We let $\Phi_L = \{(x,y) \in [\underline{x}, \overline{x}] \times \mathbb{R}_+ | x < y + R\}$ denote the subset of the strategy space in which i is an absolute favorite to lose, and let $\Phi_W = \{(x,y) \in [\underline{x}, \overline{x}] \times \mathbb{R}_+ | x > y + R\}$ denote the subset in which i is an absolute favorite to win.

3.1 Robust Comparative Statics

In this section we explore how symmetric equilibrium behavior varies with the parameters of the model; namely, n, V, and R. A symmetric equilibrium satisfies $x^* \in X^*(y^*)$ and $x^* = \frac{y^*}{n-1}$. Note that since $X^*(y)$ is defined for all $y \in \mathbb{R}_+$, we need to ensure that the candidate equilibrium value of y corresponds to a feasible strategy profile for the other players. But since $x^* \in X^*(y^*)$ implies $x^* \in [\underline{x}, \overline{x}]$, clearly, $y^* = (n-1)x^*$ is feasible.

We begin by establishing a regularity property of players' best-response correspondences. The following lemma is useful in doing so.

Lemma 1. *Let $x'' > x' \geq \underline{x}$ and $y'' > y' \geq 0$. If $x''x' < [>](y'' + R)(y' + R)$. Then,*

$$\pi(x'', y'') - \pi(x'', y') < [>] \pi(x', y'') - \pi(x', y'). \tag{2}$$

It is well-known that no CSF can be supermodular or submodular in own and rivals' efforts over its entire domain. To illustrate, suppose $n = 2$ and let P denote the CSF. If the contest is decisive (i.e., one of the two players will win the contest),[4] then for all e_1 and e_2 it holds: $P(e_1, e_2) + P(e_2, e_1) \equiv 1$. Suppose P is twice differentiable, then, $P_{12}(e_1, e_2) = -P_{12}(e_2, e_1)$. It is easy to see that this implies whenever player 1's objective function is supermodular in (e_1, e_2), then player 2's objective function is submodular, and vice versa. Yet, Lemma

[4]Under the logit CSF, when $R > 0$, the contest is not decisive. Nevertheless, a similar logic applies.

1 implies that, under the logit CSF, π is strictly submodular/supermodular over a subset of the strategy space. Specifically, π is strictly submodular on Φ_L and strictly supermodular on Φ_W.

Now, using the Implicit Function Theorem, Dixit (also see Jensen, 2016) shows that when i's rivals increase their aggregate action, i responds aggressively to defend her position (by increasing her action) when she is a favorite to win, and responds timidly (by reducing her action) when she is a favorite to lose. Using Lemma 1, our next result follows almost immediately by Topkis' Theorem,[5] and it extends this regularity property found by Dixit.

Lemma 2. *Suppose Assumption 1 is satisfied. Let $y'' > y' \geq 0$, and let $x'' \in X^*(y'')$ and $x' \in X^*(y')$. If $x''x' < (y'' + R)(y' + R)$ then $x'' \leq x'$. If $x''x' > (y'' + R)(y' + R)$ then $x'' \geq x'$.*

Lemma 2 It implies that every selection from X^* is increasing [decreasing] when it is fully contained in Φ_W [Φ_L]. Figure 1 illustrates a typical best-response. Φ_W is the region above the dashed line, $x = y + R$, and Φ_L is the region below this line. Note that in a symmetric equilibrium, $x = \frac{y}{n-1} < y+R$, which means any symmetric equilibrium occurs in Φ_L.[6] Since each selection from the best-response is decreasing in this region, and the "equilibrium line", $x = \frac{y}{n-1}$, is strictly increasing, there can be, at most, one point of intersection between the two. Thus any symmetric equilibrium must be unique. Our next result states this observation formally.

Proposition 1. *Suppose Assumption 1 is satisfied. For a fixed $n \geq 2$, if a symmetric equilibrium exists, it must be unique.*

Also observe in Figure 1 that an increase in n rotates the equilibrium line downwards. It is clear from the figure that equilibrium individual force must decrease, while the total force of i's rivals must increase. Our next result formalizes these observations, and also shows how payoffs vary with n. In what

[5]See, e.g., Theorem 3.1 in Vives (1990) or Theorem A.1 in AL.

[6]Note that if $R = 0$ and $n = 2$, this is no longer true, as the lines $x = y + R$ and $x = \frac{y}{n-1}$, coincide. We discuss this in Section 3.3.

follows, we let x_t denote the symmetric equilibrium force when the parameter of interest is equal to t, and analogously define y_t, z_t, and π_t.

Proposition 2. *Suppose Assumption 1 is satisfied. For a fixed $n \geq 2$, in a symmetric equilibrium, with $n'' > n'$,*

(i) The individual force (and effort) is decreasing in the number of players: $x_{n''} \leq x_{n'}$.

(ii) The other players' joint force is increasing in the number of players: $y_{n''} \geq y_{n'}$.

(iii) The expected per-player payoff is decreasing in the number of players: $\pi_{n''} \leq \pi_{n'}$.

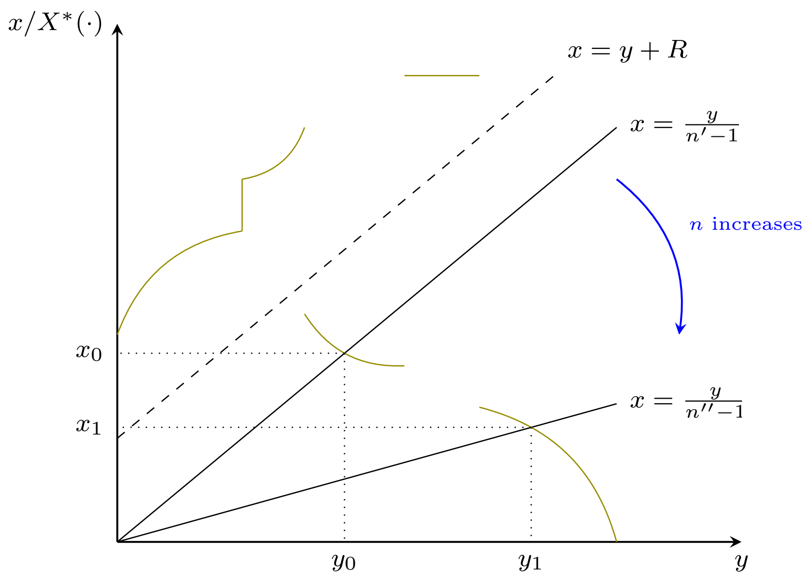

Figure 1: An example of a typical best-response correspondence (in green). Each selection from X^* must be increasing in the region above the line $x = y + R$, and decreasing in the region below this line.

Consistent with other results that invoke lattice-theoretic techniques, our findings only ensure weak monotonicity of individual forces/efforts and payoffs in n. The following example shows that strict monotonicity cannot be guaranteed in general.

Example 1. *Let* $V = 10$, $R = 0$,[7] $\underline{x} = 0$, *and* $\kappa(x) = \begin{cases} \sqrt{x} & x \leq 1 \\ (x+1)^2 - 3 & x \geq 1 \end{cases}$. *For each* $n = 2, \ldots, 10$ *there is a unique symmetric equilibrium in which all players choose* $x^* = 1$.

We next study how symmetric equilibrium behavior depends on the other parameters of the model, namely, the value of the prize, V, and the discount rate, R. Before doing so, we introduce a new parameter that may be of interest in some applications. Suppose that each player's cost function depends on a parameter, $\theta \in \mathbb{R}$. Assume that C is strictly submodular in (e, θ). That is, for all $e'' > e'$ and $\theta'' > \theta'$ assume:

$$C(e'', \theta'') - C(e', \theta'') < C(e'', \theta') - C(e', \theta')$$

If C is twice differentiable in e and θ, then the condition above is implied by $C_{12} < 0$ (equivalently, $\kappa_{12} < 0$). That is, an increase in θ strictly decreases the marginal cost of effort.

Proposition 3. *Fix* $n \geq 2$ *and suppose Assumption 1 is satisfied. Then in a symmetric equilibrium,*

(i) Individual and total forces/efforts are increasing in the value of the prize: For $V'' > V'$, $x_{V''} \geq x_{V'}$ *and* $z_{V''} \geq z_{V'}$.

(ii) Individual and total forces/efforts are increasing in the cost parameter: For $\theta'' > \theta'$, $x_{\theta''} \geq x_{\theta'}$ *and* $z_{\theta''} \geq z_{\theta'}$.

(iii) Individual and total forces/efforts are decreasing in the discount rate: For $R'' > R'$, $x_{R''} \leq x_{R'}$ *and* $z_{R''} \leq z_{R'}$.

Parts (i) and (ii) of Proposition 3 follow from the fact that π is strictly supermodular in (x, V) and (x, θ). By Topkis' Theorem, an increase in either of these parameters shifts a player's best response upward, and leads to an

[7] The main idea in this example would also hold for R strictly positive but sufficiently small.

increase in individual force/effort in any symmetric equilibrium. Figure 2 illustrates: Following an increase in V or θ, the player's best response shifts up from the green to the blue curve, and the symmetric equilibrium increases from the point E_0 to the point E_1.

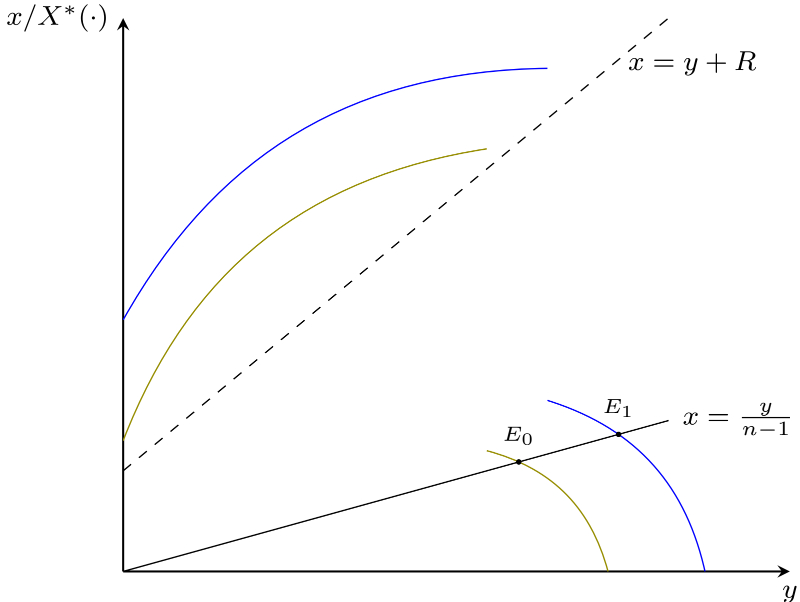

Figure 2: An illustration of the impact of an increase in V or θ on the best response correspondence. The green curve represents the best response before the parameter increase; the blue curve represents the best response after the parameter increase. Following the parameter change, the symmetric equilibrium increases from E_0 to E_1.

To understand part (iii) of Proposition 3, first note that π depends on R and y insofar as it depends on the sum, $y + R$. Adapting Lemmas 1 and 2, it is straightforward to show that, for any fixed $y \geq 0$, π is strictly submodular in (x, R) on Φ_L, and that any selection from X^* is decreasing in R whenever X^* is contained in Φ_L. So, suppose R increases from R' to R''. When a player's best response is contained in Φ_L (for both parameter values), this portion of the best response must shift down, following the increase in the parameter. Figure 3 illustrates; the green curve is the best response when $R = R'$, and the blue curve is the best response when $R = R'' > R'$. In the region between the lines, $x = y + R'$ and $x = y + R''$, the relationship between the two best-response functions is difficult to ascertain in general. In the region below the line $x = y + R'$, there is a clear ordering between the two, with the best-response shifting down following the increase in the parameter. As any symmetric equilibrium occurs below this line, we can conclude that the symmetric equilibrium force decreases.

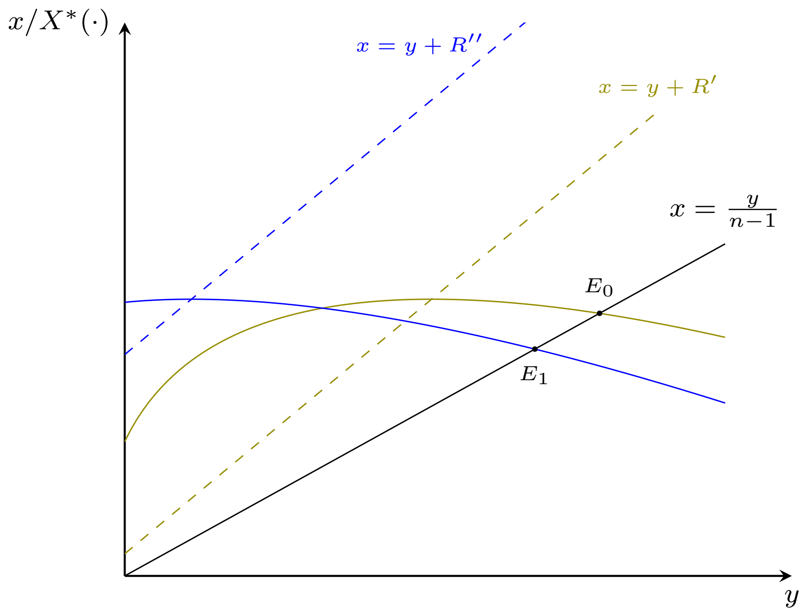

Figure 3: An illustration of the impact of a change in R on the best response. The green curve is the best response when $R = R'$; the blue curve is the best response when $R = R'' > R'$. Following an increase in R, the equilibrium decreases from E_0 to E_1.

We now explore the relationship between the parameters of the model, and equilibrium payoffs. When the contest technology exhibits decreasing returns to scale, Nti shows that equilibrium payoffs increase in V. The next example shows that this result does not hold in general. Indeed, equilibrium per-player payoffs may be strictly decreasing in the prize or the cost parameter, θ.

Example 2. *Let $n = 2$, $R = 0$,*[8] *$\underline{x} = 0$, and*

[8]The main idea in this example would also hold for R strictly positive but sufficiently small.

$$\kappa(x,\theta) = \begin{cases} \frac{x}{\theta} & x \leq 1 \\ \frac{x^\alpha}{\alpha\theta} + \frac{\alpha-1}{\alpha\theta} & x \geq 1 \end{cases}$$

If $1 \leq \frac{V\theta}{4} \leq \frac{1-\alpha}{1-2\alpha}$*, the symmetric equilibrium per-player force is* $x^* = \left(\frac{V\theta}{4}\right)^{\frac{1}{\alpha}}$*. The equilibrium per-player payoff is*

$$\pi^* = \frac{1-\alpha}{\alpha\theta} - V\left(\frac{1-2\alpha}{4\alpha}\right).$$

For $0 < \alpha < \frac{1}{2}$*,* π^* *is is strictly decreasing in* V *for fixed* θ*, and strictly decreasing in* θ *for fixed* V*. For instance, if* $\alpha = \frac{1}{3}$ *and* $\theta = 1$ *then for* $4 \leq V \leq 8$*,* $x^* = \left(\frac{V}{4}\right)^3$*, and* $\pi^* = 2 - \frac{V}{4}$*. If* $\alpha = \frac{1}{3}$ *and* $V = 1$ *then for* $4 \leq \theta \leq 8$*,* $x^* = \left(\frac{\theta}{4}\right)^3$ *and* $\pi^* = \frac{2}{\theta} - \frac{1}{4}$*.*

There are two competing forces acting on a player i's payoff following an increase in the prize. There is a direct positive effect, since the value of winning increases. But there is also an indirect negative effect, since i's rivals increase their forces. Nti shows that the direct positive effect outweighs the indirect negative effect when the contest technology exhibits decreasing returns to scale. With decreasing returns, higher levels of force are produced at greater marginal cost, which dampens the indirect effect. For technologies with increasing returns, a contrasting logic applies: Players' optimal efforts are more sensitive to changes in the prize, and the indirect effect is exacerbated.[9] As Example 2 shows, with increasing returns it may well be that the indirect effect dominates the direct effect. A similar intuition holds for changes in θ.

While the relationship between equilibrium per-player payoffs and V/θ is ambiguous, our next result shows that there is a clear relationship between payoffs and R.

Proposition 4. *Suppose Assumption 1 is satisfied. In a symmetric equilibrium, the per-player expected payoff is decreasing in the discount rate: For*

[9] This effect is evident in Example 2, as $\frac{\partial^2 x^*}{\partial\alpha\partial V} < 0$. That is, a decrease in α, which leads to more pronounced increasing returns to scale, implies that x^* is more sensitive to changes in the prize.

$R'' > R'$, $\pi_{R'} \geq \pi_{R''}$.

As with an increase in V or θ, there are two competing forces acting on equilibrium payoffs following a decrease in R. There is a direct positive effect, since a decrease in R increases the likelihood of a player winning the prize. But there is an indirect negative effect caused by the increase in the efforts of one's rivals (see Proposition 3(iii)). As it happens, the positive effect always outweighs the negative effect.

Although y is decreasing in R, the proof of Proposition 4 shows that the sum, $y+R$, is increasing in R. To see why, consider a decrease in R from R'' to R'. Figure 4 illustrates the impact of the parameter change: The green curve is (a portion of) the best response when $R = R''$, and the blue curve is the best response when $R = R' < R''$. When the parameter is R'', the total equilibrium force of i's rivals is y''. Now since X^* depends only on the sum, $y + R$, an increase in R shifts the best response outwards by a horizontal distance equal to $R'' - R'$. As x'' is a best response to y'' when $R = R''$, then x'' must also be a best response to $\hat{y} = y'' + R'' - R'$ when $R = R'$. Using the monotonicity of the best response in the region relevant for equilibrium, it is clear from the figure that any point of intersection between the blue best response and the line $x = \frac{y}{n-1}$ must occur at a value $y' \in [y'', \hat{y}]$. This means $y' + R' \leq y'' + R''$. Using this fact, it is straightforward to show that a player's equilibrium payoff must increase following a decrease in R.[10]

Summarizing

The comparative statics results provided in Propositions 2 - 4 extend results in Nti (1997). We have shown that neither decreasing returns to scale nor differentiability are essential drivers of these results. Rather, the structure imposed by the logit CSF yields payoff functions that exhibit strict increasing [decreasing] differences over Φ_W [Φ_L]. This feature drives the regularity prop-

[10] Figure 4 illustrates a scenario in which $y + R$ strictly decreases in equilibrium following the decrease in R. But this need not be the case: Lemma 2 does not preclude the possibility that $\hat{x} = \frac{\hat{y}}{n-1}$ is also a best response to $\hat{y}$ when $R = R'$ (although this would not be possible in the way we've drawn the figure).

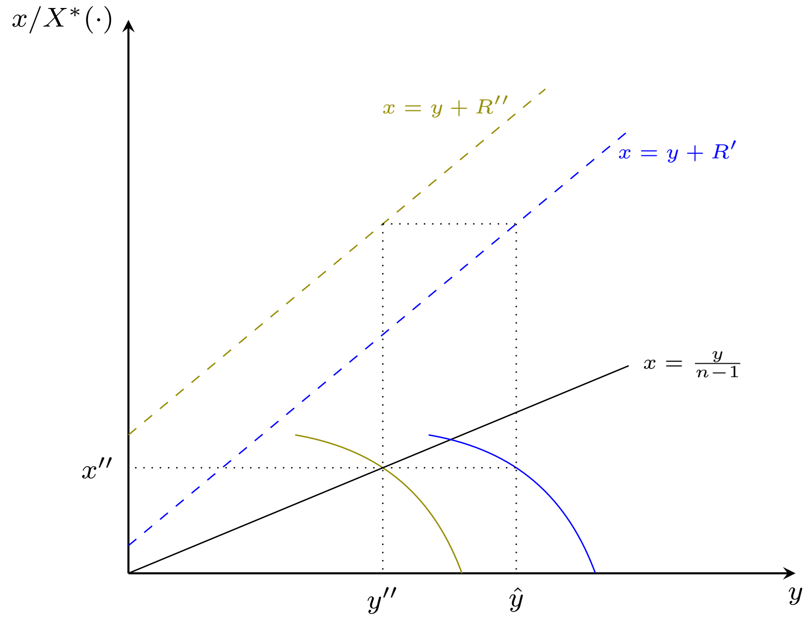

Figure 4: An illustration of the impact of a change in R on (a portion of) the best response correspondence. The green curve is the best-response when $R = R''$; the blue curve is the best response when $R = R' < R''$.

erty of the best response found by Dixit (1987), which we extend in Lemma 2. From this property, and the fact that any symmetric equilibrium occurs in Φ_L, the comparative statics follow. But Example 2 shows that the assumption of decreasing returns to scale *is* an important driver of the typical finding that equilibrium payoffs increase in the value of the prize.

3.2 Equilibrium Existence and Effects of Entry

We next turn to the question of existence, and study how aggregate behavior varies with the number of players. These results rely on the equivalence between our contest and a symmetric Cournot oligopoly with inverse demand, $\frac{V}{R+Q}$, and cost function κ. The next results are consequences of our Proposition 1, and results in AL. We strengthen Assumption 1 as follows:

Assumption 2.

(i) For all $e > 0$, $\phi(e)$ is twice continuously differentiable with $\phi'(e) > 0$.

Moreover, $\phi(0) = 0$.

(ii) For all $e \geq 0$, $C(e)$ *is twice continuously differentiable with* $C'(e) \geq 0$.

We do not wish to rule out the Tullock CSF, which is not differentiable at zero when $r < 1$. For this reason, we do not require $\phi(0)$ to be differentiable; Assumption 2 is otherwise consistent with assumptions in AL.

Noting that $z = x + y$, a player's payoff can be expressed, $\tilde{\pi}(z, y) = \frac{z-y}{R+z}V - \kappa(z-y)$. We can then think of some player i as choosing the total force, taking the total force of the other players as given. That is, player i solves, $\max\{\tilde{\pi}(z, y)|\overline{x} + y \geq z \geq y\}$. Assumption 2 ensures that $\kappa(x)$ is twice continuously differentiable for all $x > 0$. For all $z > y$, let $\Delta(z, y)$ denote the cross partial of $\tilde{\pi}$ with respect to z and y: $\Delta(z, y) = \frac{V}{(R+z)^2} + \kappa''(z-y)$. Note that Δ is defined on the lattice, $\Phi = \{(z, y) \in \mathbb{R}_+ | (n-1)\overline{x} \geq y \geq 0, y + \overline{x} \geq z > y\}$. As in AL, the sign of Δ on Φ plays a critical role in our analysis. When $\Delta >$ [<]0 on Φ, $\tilde{\pi}$ is strictly supermodular [submodular] on its domain. First we consider the case where $\Delta > 0$ on Φ.

Proposition 5. *In addition to Assumptions 1(iii) and 2, suppose* $\Delta > 0$ *on* Φ. *Then,*

(i) For each $n \in \mathbb{N}$, *there exists a unique symmetric equilibrium, and no asymmetric equilibria.*

(ii) The equilibrium total force is increasing in n*: For* $n'' > n'$, $z_{n''} \geq z_{n'}$.

Proposition 5 provides conditions under which the existence of a unique symmetric equilibrium can be guaranteed, and also shows that equilibrium total force is increasing in the number of players. The existence result follows from AL, while uniqueness follows by Proposition 1. The condition, $\Delta > 0$, limits the returns to scale of the contest technology, but also depends on the magnitudes of the discount rate and the prize. Jensen (2016) provides a result similar to Proposition 5 in an asymmetric contest under a different sufficient condition. In the special case where all players are symmetric, Jensen's Assumption 3 amounts to, $\frac{C''}{C'} \geq \frac{\phi''}{\phi'}$, which is equivalent to $\kappa'' \geq 0$. Note that

$\Delta > 0$ if $\kappa'' \geq 0$, but Δ may be strictly positive even if κ is strictly concave everywhere (see Example 3 in Gama and Rietzke, 2017).

We now consider the case where $\Delta < 0$. Our next two results are consequences of AL's Theorems 2.5 and 2.6, respectively.

Proposition 6. *In addition to Assumptions 1 and 2, suppose $\Delta < 0$ on Φ. Then, for any $n \in \mathbb{N}$,*

(i) For any $m < n$, if a symmetric equilibrium exists in the m-player contest (with individual force x_m, say) then the following configuration constitutes an equilibrium for the n player contest: Each of any m players chooses force x_m while the remaining $n - m$ players exert zero effort. In particular, an n-player equilibrium always exists in which one player chooses the optimal single-player effort and the other $n-1$ players choose zero effort.

(ii) A unique symmetric equilibrium exists if, for each $y \in [0, \overline{y}]$, $\pi(\cdot, y)$ is strictly quasiconcave.

(iii) No other equilibrium other than those described in parts (i) and (ii) can exist.

When $\Delta < 0$ over its domain, several asymmetric equilibria may exist;[11] moreover, there always exists an equilibrium in which $n-1$ players are inactive. An equilibrium with a single active player is not standard in contests. Note that the condition, $\Delta < 0$ requires $\kappa'' < 0$ and $R > 0$. It should be clear that an equilibrium with only one active player could never exist if $R = 0$.

Our next result shows how total equilibrium forces vary with the number of players when $\Delta < 0$.

Proposition 7. *In addition to Assumptions 1 and 2, suppose $\Delta(z, y) < 0$ on Φ. Then,*

[11] Perez-Castrillo and Verdier (1992) and Cornes and Hartley find a similar equilibrium structure under the Tullock CSF with $r > 1$. Chowdhury and Sheremeta (2011) also show the potential for multiple asymmetric equilibria under the Tullock CSF when the size of the prize varies with efforts.

(i) Under the hypothesis of Proposition 6(i), all the asymmetric equilibria for all $m < n$ are invariant in the number of players n.

(ii) Under the hypothesis of Proposition 6(ii), the total equilibrium force, z_n, is decreasing in n.

Proposition 7(i) clearly holds since, in the asymmetric equilibria with $m < n$, an additional player exerts no effort. Proposition 7(ii) contrasts much of the work in contests under the logit CSF.[12] The intuition is akin to that of a natural monopoly. The condition $\Delta < 0$ implies strong increasing returns to scale. Limiting entry allows a player to take full advantage of her increasing returns. Example 4 in Gama and Rietzke (2017) illustrates our finding.

Related to Proposition 7, Baye et al. (1993) show that total effort increases if the player with the highest value is excluded in an asymmetric all-pay auction. Intuitively, the high-value player discourages effort from the other players; removing this player "levels the playing field", and encourages greater effort among the remaining players. In contrast to the Exclusion Principal, our result applies in a symmetric contest, so there is no scope for leveling the playing field. However, with increasing returns to scale, the addition of another player can have a strong discouragement effect.[13]

3.3 The case, R = 0

Since much of the work in the contest literature utilizing the logit CSF assumes $R = 0$, in this section we briefly discuss this case. The formal results discussed herein can be found in Gama and Rietzke (2017). Our results in Section 3.1 rely heavily on the fact that any symmetric equilibrium occurs in Φ_L. But when $R = 0$ and $n = 2$, this is no longer true. In this case, multiple symmetric equilibria may exist (at most 2); but uniqueness is restored if $n > 2$ or κ is

[12] One exception is Amegashie (1999). Under the Tullock CSF, Amegashie shows that total effort may be decreasing in n, when a player's prize includes a fixed component, and a variable component, which increases linearly in a player's effort.

[13] The choice of CSF plays a critical role in driving the Exclusion Principle. For instance, when marginal cost is constant, it is well-known that the result does not hold under the Tullock CSF (See, e.g., Fang, 2002; Matros, 2006; Menicucci, 2006).

differentiable. Regardless, our comparative statics in Section 3.1 continue to hold if one replaces “in a symmetric equilibrium” with “in the smallest and largest symmetric equilibria”. Proposition 5 also holds when $R = 0$. However, the condition $\Delta < 0$, which is relevant for Propositions 6 and 7, requires $R > 0$.

4 Conclusion

In this paper, we established a number of robust comparative statics results in a contest under the logit CSF with minimal restrictions on the primitives. We furthermore shed new light on the issues of equilibrium existence and uniqueness. Our results help to clarify the relevance of the structure typically imposed in this model. The main innovation of this paper is the application of lattice-theoretic techniques, which have not previously been applied in contests. Utilizing these tools, we generalized the regularity property found by Dixit on the shape of a player’s best response, and used this property to establish several comparative statics. Our approach sheds light on how lattice-theoretic techniques can be applied in contests, which should help guide future research in developing more robust conclusions.

5 Appendix

Proof of Lemma 1

Let $x'' > x' \geq \underline{x}$ and $y'' > y' \geq 0$. Let $\tilde{y}'' = y'' + R$ and $\tilde{y}' = y' + R$. The reader can easily verify that expression (2) is equivalent to

$$\frac{x''}{x'' + \tilde{y}''} - \frac{x''}{x'' + \tilde{y}'} < [>] \frac{x'}{x' + \tilde{y}''} - \frac{x'}{x' + \tilde{y}'},$$

which holds if and only if $x''x' < [>] \tilde{y}''\tilde{y}'$. □

Proof of Lemma 2

We will show the first statement in the lemma; the proof of the second statement is analogous. Let $y'' > y' \geq 0$, $x'' \in X^*(y'')$ and $x' \in X^*(y')$. Suppose $x''x' < (y''+R)(y'+R)$. Proceed by contradiction: Suppose, contrary to the lemma, $x'' > x'$. Then by Lemma 1,

$$0 \leq \pi(x'', y'') - \pi(x', y'') < \pi(x'', y') - \pi(x', y') \leq 0,$$

where the l.h.s. inequality follows since $x'' \in X^*(y'')$, and the r.h.s. inequality follows since $x' \in X^*(y')$.[14] We have a contradiction; hence it must be that $x'' \leq x'$. □

Proof of Proposition 1

Fix $n \geq 2$. We show that if a symmetric equilibrium exists, it must be unique. Let x_1 and x_2 be two symmetric equilibrium levels of force, and let let $y_i = (n-1)x_i$, $i = 1, 2$. Proceed by contradiction, and suppose that these two equilibria are distinct; in particular, suppose $x_1 > x_2$; equivalently, $y_1 > y_2$. Note that for $i = 1, 2$, x_i and y_i satisfy: $x_i \in X^*(y_i)$ and $x_i = \frac{y_i}{n-1}$. But since $\frac{y_i}{n-1} \leq y_i$, it must hold that $x_i < y_i + R$ for $i = 1, 2$; hence, $x_1 x_2 < (y_1 + R)(y_2 + R)$. Since $y_1 > y_2$ (by assumption), Lemma 2 implies $x_1 \leq x_2$, which yields a contradiction. Therefore, the symmetric equilibrium must be unique. □

Proof of Proposition 2

Parts (i)-(ii)

Fix $n'' > n'$, and suppose that a symmetric pure-strategy equilibrium exists in the n'' and n' player contests. From the arguments in the first part of this proof, we know that the symmetric equilibrium must be unique. Let x'' (x') denote the individual equilibrium individual force when the contest has n'' (respectively, n') players; let $y'' = (n''-1)x''$ and $y' = (n'-1)x'$.

[14] The choice set, $[\underline{x}, \overline{x}]$, is independent of y, so x' (x'') is certainly feasible when others' joint force is y'' (y').

We first show part (ii). First note that if $y' \leq (n''-1)\underline{x}$, then as feasibility requires $y'' \geq (n''-1)\underline{x}$, it follows that $y'' \geq y'$. Then suppose $y' > (n''-1)\underline{x}$. We will show that there cannot be a symmetric equilibrium with $y < y'$ when $n = n''$. Fix $y_0 \in [(n''-1)\underline{x}, y')$, and let $x_0 \in X^*(y_0)$. Clearly if $x_0 \geq y_0 + R$ then $x_0 > \frac{y_0}{n''-1}$. If $x_0 < y_0 + R$, then Lemma 2 implies $x_0 \geq x'$. But since $n'' > n'$ and $y' > y_0$ it holds, $x' = \frac{y'}{n'-1} > \frac{y'}{n''-1} > \frac{y_0}{n''-1}$; thus, $x_0 > \frac{y_0}{n''-1}$. We have now established that for all $y \in [(n''-1)\underline{x}, y')$, $x \in X^*(y)$ implies $x > \frac{y}{n''-1}$; thus, there cannot be a symmetric equilibrium with $y < y'$ when $n = n''$. It follows that $y'' \geq y'$. This establishes part (ii).

Next, we show $x'' \leq x'$. We have already shown that $y'' \geq y'$. If $y'' = y'$ then $n'' > n'$, implies $x'' = \frac{y''}{n''-1} < \frac{y'}{n'-1} = x'$. If $y'' > y'$ then since $x'' < y'' + R$ and $x' < y' + R$, Lemma 2 implies $x'' \leq x'$. Since individual force decreases following the increase in n, clearly individual effort decreases as well. This establishes part (i).

Part (iii)

Let π'' (π') denote the equilibrium individual expected payoff in the contest with n'' (respectively, n') players where $n'' > n'$. We have the following string of inequalities:

$$\begin{aligned}
\pi' &= \frac{x'}{R + x' + y'}V - \kappa(x') \\
&\geq \frac{x''}{R + x' + y'}V - \kappa(x'') \\
&\geq \frac{x''}{R + x'' + y''}V - \kappa(x'') \\
&= \pi''
\end{aligned}$$

The first inequality holds by definition of x'. The second inequality follows since, as we showed in part (ii), $y'' \geq y'$. This establishes part (iii) and the proposition. □

Proof of Proposition 3

We prove parts (i) and (ii) jointly. Fix n, and let t denote either the parameter V or θ. It is easily verified that π is strictly supermodular in t and x. By Topkis' Theorem (see, e.g. Topkis, 1978, or Theorem A.1 in AL), it follows that any selection, $r(\cdot)$, from X^* is increasing in t for each y. It is clear that this implies that any intersection between a player's reaction curve and the line $x = \frac{y}{n-1}$ must lie further from the origin following an increase in t. Since individual force increases, then individual effort and total force/effort must increase, as n is fixed. This proves parts (i) and (iii).

Next, we show part (iii). Let $R'' > R'$, and suppose a symmetric equilibrium exists for both parameter values. Let x'' and x' denote the symmetric equilibrium individual force when the parameter is R'', respectively R'. Let $y'' = (n-1)x''$ and $y' = (n-1)x'$. Since π depends on y and R insofar as it depends on the sum, $y + R$, X^* depends only on this sum. We write $X^*(y+R)$ to denote the set of best-replies to y when the parameter is R. Note that our result follows immediately if $x'' = \underline{x}$; so, assume $x'' > \underline{x}$; equivalently, $y'' > \underline{y}$.

Let $\hat{y} = R'' - R' + y'' > y''$. By construction, $\hat{y} + R' = y'' + R''$; therefore, $X^*(\hat{y}+R') = X^*(y''+R'')$. By definition of x'', this means $x'' \in X^*(\hat{y}+R')$. We now show that there cannot be a symmetric equilibrium in which $y < y''$ when $R = R'$. Fix $y_0 \in [\underline{y}, y'')$, and let $x_0 \in X^*(y_0 + R')$. Clearly, if $x_0 > y_0 + R'$ then $x_0 > \frac{y_0}{n-1}$. If $x_0 \leq y_0 + R'$, then since $x'' \leq y'' < \hat{y} + R'$, it holds $x_0 x'' < (y_0 + R')(\hat{y} + R')$. As, $x_0 \in X^*(y_0 + R')$, $x'' \in X^*(\hat{y} + R')$, and $\hat{y} > y'' > y_0$, Lemma 2 implies $x_0 \geq x'' = \frac{y''}{n-1} > \frac{y_0}{n-1}$.

We have now shown that for all $y \in [\underline{y}, y'')$, $x \in X^*(y+R')$ implies $x > \frac{y}{n-1}$; thus, there cannot exist a symmetric equilibrium in which $y < y''$ when $R = R'$. It therefore must be that $y' \geq y''$; equivalently, $x' \geq x''$. Since individual force increases following a decrease in R, then individual effort and total force/effort must increase, as n is fixed. This establishes part (iii) and the proposition. $\square$

Proof of Proposition 4

Let $R'' > R'$ and suppose that a symmetric equilibrium exists for both parameter values. Let x'' (x') denote the equilibrium per-player force when the parameter is R'' (R'). Let $y'' = (n-1)x''$ and $y' = (n-1)x'$. Finally, let $X^*(y+R)$ denote the best response to y when the parameter is R.[15]

Let $\hat{y} = y'' + R'' - R' > y''$. As we showed in the proof of Proposition 3(ii), it must be that $x'' \in X^*(\hat{y} + R')$. Moreover, since $x'' = \frac{y''}{n-1} < \frac{\hat{y}}{n-1} < \hat{y} + R'$, Lemma 2 implies that for all $y > \hat{y}$, if $x \in X^*(y + R')$ and $x < y + R'$, then $x \leq x'' < \frac{y}{n-1}$. Thus, there cannot exist a symmetric equilibrium with $y > \hat{y}$ when $R = R'$. Therefore, $y' \leq \hat{y}$, which implies $y' + R' \leq \hat{y} + R' = y'' + R''$. Next, let π'' (π') denote the equilibrium payoff to a player when the parameter is R'' (respectively, R'). It holds,

$$\begin{aligned}
\pi' &= \frac{x'}{x' + y' + R'}V - \kappa(x') \\
&\geq \frac{x''}{x'' + y' + R'}V - \kappa(x'') \\
&\geq \frac{x''}{x'' + y'' + R''}V - \kappa(x'') \\
&= \pi''
\end{aligned}$$

The first inequality follows by definition of x'; the second follows since $y'' + R'' \geq y' + R'$. This establishes the proposition. □

Proof of Propositions 5-7

The contest is equivalent to a symmetric Cournot oligopoly with inverse demand function $\tilde{P}(z) = \frac{V}{R+z}$, and cost function, $\tilde{C}(x) = \kappa(x)$. The existence of a symmetric equilibrium, and non-existence of any asymmetric equilibria established in Proposition 5 follows from Theorem 2.1 in AL. The uniqueness of this equilibrium follows from Proposition 1. Part (ii) of Proposition 5 follows

[15] Recall from the proof of Proposition 3(ii) that a player's best response depends on y and R insofar as it depends on the sum, $y + R$.

immediately by Theorem 2.2(b) in AL. Propositions 6 and 7 follow by AL's Theorems 2.5 and 2.6, respectively. □

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