Contests on Networks

Alexander Matros[*] David Rietzke[†]

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Abstract

We develop a model of contests on networks. Each player is "connected" to a set of contests and exerts a single effort to increase the probability of winning each contest to which she is connected. We characterize equilibria under the Tullock contest success function and explore how behavior depends on the pattern of interactions. Additionally, we show that many well-known results from the contest literature can be obtained by varying the structure of the network. Our framework has a broad range of applications, including research and development, advertising, and research funding.

Keywords: Network Games, Contests, Bipartite Graph, Exclusion Principle, Tullock Contest
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1 Introduction

In recent years, economists have recognized the importance of understanding how the structure of interactions affects economic behavior, which has led to

[*]Moore School of Business, University of South Carolina and Lancaster University Management School.

[†]Lancaster University Management School.

the development of research on networks. The importance of this field of research is unquestionable, due to the broad applicability of these models in many economically relevant settings; to name a few: job search and employment dynamics (Calvó-Armengol and Jackson, 2004; Calvó-Armengol, 2004), the provision of public goods (Bramoullé and Kranton, 2007; Bramoullé et al., 2014), collaboration/research and development (Goyal and Moraga-Gonzalez, 2001; Goyal and Joshi, 2003), and criminal activity (Calvó-Armengol and Zenou, 2004; Ballester et al., 2006). Jackson and Zenou (2014) provide a comprehensive overview of the literature on network games, and emphasize three approaches researchers have taken to study such games:[1] (1) games of strategic complements and substitutes; (2) games with linear best-replies; and (3) settings with an uncertain pattern of interactions.[2] These three approaches have proved fruitful in allowing researchers to understand how the underlying pattern of interactions affects behavior.

In this paper, we study a new class of network games: contests on networks.[3] Our model consists of a set of players and a set of contests, which form a bipartite graph (or network).[4] Each player competes in contests to which she is connected by exerting a single effort. The probability that a player wins a contest is determined by the popular Tullock contest success function (CSF).[5] Our interest here is to understand how the pattern of interactions affects behavior.

Our model has a number of applications including, for example, centralized R&D decisions by multinational firms (MNFs). Prizes are commonly used tools to encourage R&D activity,[6] and contests can be used to model both

[1] See also Jackson (2008) and Bramoullé and Kranton (2016).

[2] Examples of models with an uncertain pattern of interactions include Jackson and Yariv (2007), Galeotti et al. (2010).

[3] For relevant surveys of the contest literature see Nitzan (1994), Congleton et al. (2008), and Konrad (2009).

[4] A bipartite graph is a graph in which the vertices may be partitioned into two disjoint subsets; the edges connect the vertices from these subsets.

[5] See Tullock (1980).

[6] Innovation inducement prizes were a central feature of the Obama Administration's efforts to stimulate American innovation as part of the Recovery Act of 2009. Since 2010, more than 800 inducement prizes have been offered by federal agencies in areas ranging from

explicit R&D contests or patent races (e.g. Che and Gale, 2003; Baye and Hoppe, 2003). To take advantage of economies of scale and scope, historically, R&D activity within MNFs has tended to be centralized, and undertaken at the corporate level (Gassmann and Von Zedtwitz, 1999). In this interpretation, the firm chooses a single level of R&D effort, the benefits of which are then realized by each branch of the firm. Our model could also be interpreted in the context of a national advertising campaign by a geographically dispersed franchised firm. In this context, each firm chooses a level of expenditure on a national advertising campaign, which increases the share of the market each franchise expects to capture. Finally, one might also interpret our model in the context of research funding. In this setting, researchers exert effort on a project proposal, which they then submit to various funding agencies to increase their chances of receiving funding for their project.

The main contributions of this research are twofold. First, we contribute to the literature on networks by analyzing a new class of network games. We characterize equilibrium behavior in terms of the underlying network characteristics and study how these characteristics influence behavior. Second, we contribute to the literature on contests by establishing connections between several important observations in the contest literature and different network structures in our setting. That is, we provide a unified framework, within which many well-known results in the contest literature can be obtained by simply varying the structure of the network.

We describe a class of "quasiregular" networks, on which players behave *as if* competing in a single-prize asymmetric contest.[7] This equivalence brings analytical advantages, since behavior in the network game can be understood by studying a simpler single-prize contest. In particular, we are able to provide closed-form solutions for players' efforts given in terms of network characteristics. This then allows us to assess how equilibrium efforts are influenced by the pattern of interactions.

national defense to education. See, `http://challenge.gov/about`.

[7]See Hillman and Riley (1989), Nti (1999, 2004), Stein (2002), and Matros (2006) for the characterization of equilibrium in asymmetric contests under the Tullock CSF.

A striking observation in the contest literature is the Exclusion Principle, first introduced by Baye et al. (1993). The Exclusion Principle states that total equilibrium effort may increase if the most competitive (the highest value) player is excluded from the contest. Intuitively, the high-value player has a "discouragement effect" on less competitive players. Excluding the high-value player "levels the playing field", which results in a more competitive contest and leads the remaining players to exert higher effort – enough to offset the loss of the high-value player's effort. It is a common finding in the contest literature that the Exclusion Principle holds only under the all-pay auction (APA) CSF, and does not apply to the Tullock CSF (see, for example, Fang, 2002; Matros, 2006; Menicucci, 2006). The reason is that the Tullock CSF introduces a significant amount of noise in determining the outcome of the contest as compared to the APA CSF. As a result, competition is softer, and the discouragement effect is less pronounced. In this paper, we derive a new exclusion principle, given in terms of network structures, which also applies under the Tullock CSF.

Our study adds to a growing literature that combines contests and networks. Franke and Öztürk (2015) and Huremovic (2016) study bilateral conflicts in which players may compete for several prizes and each player chooses a vector of efforts. Xu and Zhou (2018) study a contest under the logit CSF and allow for arbitrary network structures. The authors employ techniques from Variational Inequality to explore existence/uniqueness of equilibrium, and generate some comparative statics. Our model differs from these three studies in that, in our model, each player chooses a single effort; in these models, each player chooses a vector of efforts (one effort for each contest in which the player competes).

König et al. (2017) studies a setting in which players, competing for a single prize, are linked as enemies or allies via a network structure. The authors apply their results to provide insights on the Second Congo War. Kovenock and Roberson (2018) study the attack and defense of targets, which are connected via a network structure. In this model, the attacker's objective is to disconnect the network, while the defender's objective is to maintain network connectivity.

Marinucci and Vergote (2011) and Grandjean et al. (2016) study a model of network formation in an all-pay auction, and a Tullock contest, respectively. In these models, players compete for a single prize, but the value of the prize to each player depends on the number of links she forms. Jackson and Nei (2015) study the interaction between networks of alliances, international trade, and conflict. They find that their theoretical results are consistent with observed historical patterns.

The remainder of the paper is organized as follows. In Section 2.1 we introduce the model. In Section 2.2 we provide a characterization of equilibrium behavior for arbitrary networks. In Section 3 we introduce the class of quasiregular networks and provide several examples. In Section 4 we provide closed-form expressions for equilibrium efforts on quasiregular networks and explore how the underlying network structure affects behavior. In Section 5 we show the connection between our results and the existing contest literature. Concluding remarks are given in Section 6.

2 The Model and General Approach

2.1 The Model

There are N players and M contests; let $\mathcal{N} = \{1, \ldots, N\}$ denote the set of players, and $\mathcal{M} = \{1, \ldots, M\}$ denote the set of contests. Each player i is risk neutral and is characterized by a vector $\mathbf{g}_i \equiv (g_{i1}, \ldots, g_{iM})$ where $g_{im} = 1$, if player i competes in contest m, and $g_{im} = 0$ otherwise. If player i wins contest m, then she receives a prize $V > 0$, which we normalize to $V = 1$, without loss of generality. Player i chooses a single effort, $x_i \geq 0$, to increase her probability of winning each contest in which she competes.

The network structure can be represented by a bipartite graph - a graph in which the vertices can be separated into two disjoint subsets, and each edge connects the vertices from these disjoint subsets. In our setting, the two disjoint subsets are the set of players and the set of prizes; the edges indicate in which contest(s) each player competes. Figure 1 illustrates this bipartite structure of the contest network. Player 1, for example, competes in two contests; in contest 1, she competes with player 2, while in contest 4, she competes with player 3.

The network structure is summarized by the $N \times M$ biadjacency matrix:

$$\mathcal{G} = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1M} \\ g_{21} & g_{22} & \cdots & g_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} & g_{N2} & \cdots & g_{NM} \end{pmatrix}.$$

For the network structure in Figure 1:

$$\mathcal{G} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Now, we extend the classical Tullock contest to our environment. Let $\mathbf{x}_{-i} \in \mathbb{R}_+^{N-1}$ denote the vector of efforts chosen by all players other than player i. If $g_{jm}x_j > 0$ for some player j and prize m, then the probability that player i wins prize m is given by the following contest success function:

$$p_{im}(x_i, \mathbf{x}_{-i}) = \frac{g_{im}x_i}{\sum_{j=1}^{N} g_{jm}x_j}.$$

If, for some contest, m, $g_{jm}x_j = 0$ for all j then, we assume each player competing in the contest is equally likely to win; i.e., $p_{im}(0, \mathbf{x}_{-i}) = \frac{g_{im}}{\sum_{j=1}^{N} g_{jm}}$. The expected payoff to player i is

$$\pi_i(x_i, \mathbf{x}_{-i}) = \frac{g_{i1}x_i}{\sum_{j=1}^{N} g_{j1}x_j} + \cdots + \frac{g_{iM}x_i}{\sum_{j=1}^{N} g_{jM}x_j} - x_i. \quad (1)$$

Equation (1) says that the expected payoff to player i is just the sum of the expected payoffs across contests in which she competes. Player i takes $\mathbf{x}_{-i}$ as given and chooses $x_i \geq 0$ to maximize $\pi_i(x_i, \mathbf{x}_{-i})$.

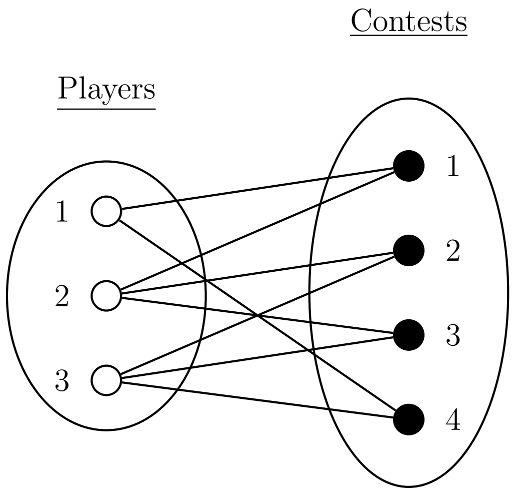

Figure 1: The contest network as a bipartite graph.

2.2 General Approach

Here, we analyze equilibrium behavior for arbitrary network structures. At an interior solution, the first-order condition for player i is:

$$\frac{\partial \pi_i(x_i, \mathbf{x}_{-i})}{\partial x_i} = g_{i1} \frac{\left(\sum_{j=1}^N g_{j1} x_j\right) - g_{i1} x_i}{\left(\sum_{j=1}^N g_{j1} x_j\right)^2} + \cdots + g_{iM} \frac{\left(\sum_{j=1}^N g_{jM} x_j\right) - g_{iM} x_i}{\left(\sum_{j=1}^N g_{jM} x_j\right)^2} - 1 = 0,$$

or

$$g_{i1} \frac{S_1 - x_i}{S_1^2} + \cdots + g_{iM} \frac{S_M - x_i}{S_M^2} = 1,$$

where $S_m = \sum_{j \in \mathcal{N}} g_{jm} x_j$ is total effort in contest $m = 1, ..., M$. Solving for x_i,

$$x_i = \frac{\left(\frac{g_{i1}}{S_1} + \cdots + \frac{g_{iM}}{S_M}\right) - 1}{\left(\frac{g_{i1}}{S_1^2} + \cdots + \frac{g_{iM}}{S_M^2}\right)}. \tag{2}$$

Suppose that $n \leq N$ players are active, then

$$\frac{g_1 - 1}{S_1} + \cdots + \frac{g_M - 1}{S_M} = n, \tag{3}$$

where $g_m = \sum_{i \in \mathcal{N}} g_{im}$ is the degree of contest m. In summary:

Theorem 1. *Suppose that an equilibrium exists in Contests on a Network. If there are $n \leq N$ active players in the equilibrium, then*
(1) each active player, i, exerts effort according to expression (2)*;*
(2) equation (3) *holds;*
(3) player j is inactive if and only if $\sum_{m \in \mathcal{M}} \frac{g_{jm}}{S_m} \leq 1$.

Theorem 1 characterizes equilibrium behavior for arbitrary network structures. In the next section, we introduce some additional structure on the network, which will allow us to obtain closed-form solutions for equilibrium efforts.

3 Quasiregular Networks

Let $d_i = \sum_{m \in \mathcal{M}} g_{im}$ denote the degree of player i. Let $K_m(\gamma)$ denote the number of players with degree γ who compete in contest $m \in \mathcal{M}$:

$$K_m(\gamma) = \sum_{i \in \mathcal{N}} g_{im} \mathbb{1}(d_i = \gamma),$$

where $\mathbb{1}(\cdot)$ is an indicator function.

Definition 1. *We say that the network is quasiregular if $K_m(\gamma) = K(\gamma)$ and $\sum_{i \in \mathcal{N}} g_{im} \geq 2$ for each $m \in \mathcal{M}$ and $\gamma \in \{1, \ldots, M\}$.*

Definition 1 says that the number of players competing in a contest with degree γ is the same across contests. Note that, in a quasiregular network, each contest has the same degree: For each m, $\sum_{i \in \mathcal{N}} g_{im} = \sum_{\gamma=1}^{M} K(\gamma) = g$, while the degree of each player can be different. Quasiregularity requires $g \geq 2$, which ensures that there is at least *some* competition in each contest. For quasiregular networks, the following link property must be satisfied:

$$\sum_{i \in \mathcal{N}} d_i = Mg \tag{4}$$

The left-hand side of (4) is the number of links from players to prizes; the right-hand side is the number of links from prizes to players; obviously, these two numbers must be equal.

Figure 2 provides an illustration of a quasiregular network. For this network, each contest has 4 players: 1 of these players has degree 1, 2 players have degree 2, and 1 player has degree 3. Thus, $g = 4$, $K(1) = 1$, $K(2) = 2$, and $K(3) = 1$.

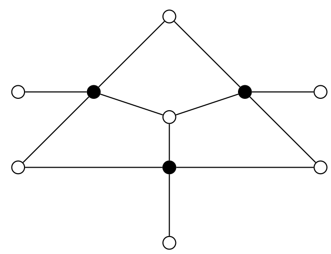

Figure 2: An example of a quasiregular network. Players are represented by hollow nodes; contests are represented by solid nodes.

3.1 Examples of Quasiregular Networks

In this section we present several special classes of quasiregular networks.

Biregular Networks

A bipartite network, $\mathcal{G}$, is *biregular* if each pair of nodes in the disjoint subsets of $\mathcal{G}$ have the same degree. Since quasiregularity implies that each contest has the same degree, a biregular network is a special case of a quasiregular network in which each player also has the same degree: $d_i = d_j = d$ for some integer, d. Formally,

Definition 2. *Biregular Network*
Let $\mathcal{G}$ be a quasiregular network with prize degree, g. $\mathcal{G}$ is biregular if, for some $d \geq 1$, $K(d) = g$, and $K(\gamma) = 0$ for $\gamma \neq d$.

A biregular network with N players, M prizes, contest degree, g, and player degree, d, can be summarized by $[N, d; M, g]$. For biregular networks, the link property, (4) collapses to,

$$Nd = Mg. \tag{5}$$

Two special cases of biregular networks are *complete networks*, and *circle networks*.[8] In a complete network, each player competes for all prizes: $d = M$, and $g = N$. In a circle network, each player has degree two ($d = 2$), each contest has degree two ($g = 2$), but no two contests have the same two participants.

Example 1 describes all biregular networks for the case of $N = 3$; Figure 3 illustrates the corresponding networks.

Example 1. *Suppose that there are $N = 3$ players.*

- *If $M = 1$, then there is a unique biregular network, which is the complete network: $g = 3$ and $d = 1$.*
- *If $M = 2$, then there is a unique biregular network, which is the complete network: $g = 3$ and $d = 2$.*
- *If $M = 3$, then there are two biregular networks:*
  - *i. A circle network where $g = d = 2$.*
  - *ii. A complete network where $g = d = 3$.*

Star Networks

In a *star network* there are $N = M + 1$ players: M "periphery players" and 1 "central player". Each periphery player competes in a single contest, while the central player competes in all M contests. A star network is a special case of a quasiregular network in which each contest has two participants, one with degree M, and the other with degree 1. Formally,

[8] Circle networks are also known as cycle graphs or ring networks.

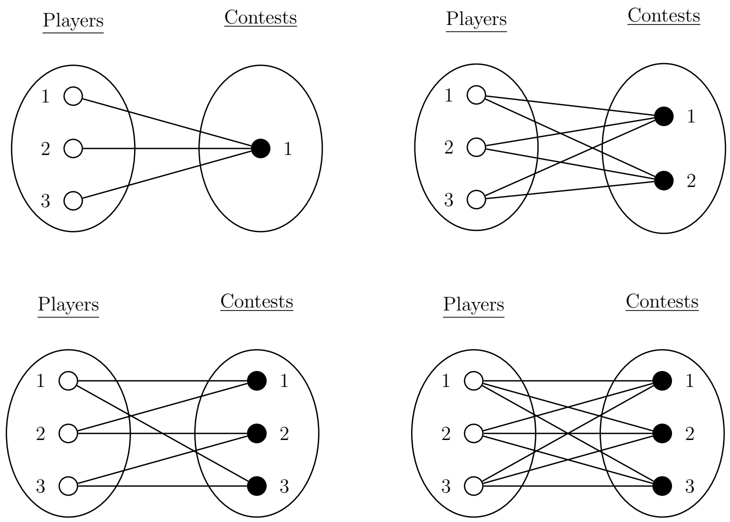

Figure 3: Illustration of the four networks described in Example 1. For all four networks, $N = 3$. Top left: A complete network with $M = 1$, $g = 3$, $d = 1$; Top right: A complete network with $M = d = 2$, $g = 3$; Bottom left: A circle network with $M = 3$, $g = d = 2$; Bottom right: A complete network with $M = g = d = 3$.

Definition 3. *Star Network*

Let $\mathcal{G}$ be a quasiregular network. $\mathcal{G}$ is a star if $M \geq 2$, $K(1) = K(M) = 1$ and $K(\gamma) = 0$ for $\gamma \notin \{1, M\}$.

Figure 4 illustrates a star network where $M = 5$.

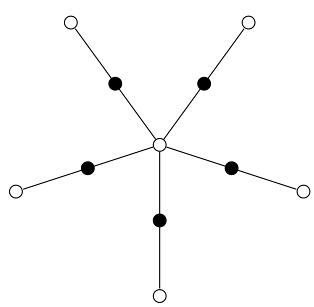

Figure 4: A star network with $M = 5$. Players are represented by hollow nodes; contests are represented by solid nodes.

Hybrid Networks

An *M-hybrid* network is a quasiregular network, which shares features of a biregular network and a star network. Specifically, let $3 \leq g \leq N$, and consider a biregular network, $\tilde{\mathcal{G}}$, summarized by $[N-1, d; M, g-1]$. Next, consider adding a player with degree, M, to this network. We call the resulting structure an M-hybrid. We call $\tilde{\mathcal{G}}$ the "underlying network", and refer to the players in this network as the "underlying players". We call the additional player with degree, M, the "hybrid player" of $\mathcal{G}$. Formally,

Definition 4. *M-Hybrid Network*
Let $3 \leq g \leq N$, and let $\tilde{\mathcal{G}}$ be a biregular network summarized by $[N-1, d; M, g-1]$. The N-player network, $\mathcal{G}$, formed by adding a player with degree M to this biregular network is called an M-hybrid with an underlying network, $\tilde{\mathcal{G}}$.

Note that for M-hybrid networks, the link property, (4), implies,

$$(N-1)d = M(g-1).$$

Figure 5 illustrates a 6-hybrid network where the underlying network is a 6-player circle.

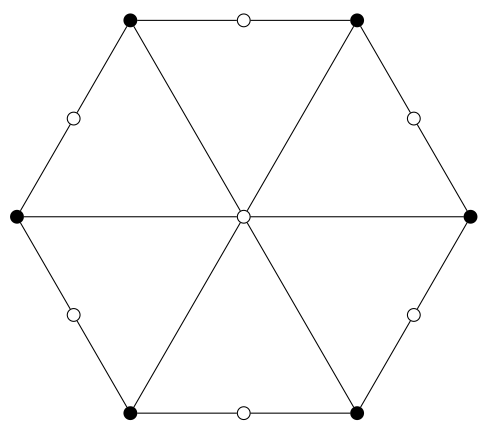

Figure 5: A 6-hybrid network. The underlying network is a 6-player circle, and the hybrid player is the center node. Players are represented by hollow nodes; contests are represented by solid nodes.

4 Results

In this section, we characterize equilibrium behavior and explore how behavior depends on the pattern of interactions. We then connect our results to the existing contest literature.

In what follows, it will be useful to have a measure that summarizes the degree of asymmetry in the network. One such measure is the ratio of the arithmetic mean of players' degrees to the harmonic mean of players' degrees. Let $\mathbf{d} = (d_1, \ldots, d_N)$ denote the vector of player degrees. Let $A(\mathbf{d}) = \frac{1}{N}\sum_{i\in\mathcal{N}} d_i$ denote the arithmetic mean of player degrees, and let $H(\mathbf{d}) = \frac{N}{\sum_{i\in\mathcal{N}}\frac{1}{d_i}}$ denote the harmonic mean of player degrees. Let $a(\mathbf{d})$ denote the ratio of these two means:

$$a(\mathbf{d}) = \frac{A(\mathbf{d})}{H(\mathbf{d})}.$$

Note that $a(\cdot) \geq 1$. For a biregular network, in which each player has the same degree, it holds, $a(\mathbf{d}) = 1$. If $d_i \neq d_j$ for some i and j then $a(\mathbf{d}) > 1$. Moreover, $a(\mathbf{d})$ is increasing in the the extent of asymmetry in the network in the following sense: Consider two networks, $\mathcal{G}_1$ and $\mathcal{G}_2$, and let $\mathbf{d}_j$ denote the vector of player degrees in network $\mathcal{G}_j$. Suppose the player degrees in $\mathcal{G}_1$ are a mean-preserving spread of the degrees in $\mathcal{G}_2$; then $A(\mathbf{d}_1) = A(\mathbf{d}_2)$. But, by properties of the harmonic mean, $H(\mathbf{d}_1) < H(\mathbf{d}_2)$. So, $a(\mathbf{d}_1) > a(\mathbf{d}_2)$. In the next example, we compute $a(\mathbf{d})$ for biregular and star networks.

Example 2. *Let $\mathcal{G}$ be a quasiregular network with N players, M contests, and prize degree, g.*

- *If $\mathcal{G}$ is biregular, then each player has the same degree, d. In this case, $A(\mathbf{d}) = d$, $H(\mathbf{d}) = \frac{N}{\frac{N}{d}} = d$, and $a(\mathbf{d}) = 1$.*
- *If $\mathcal{G}$ is a star, then there are $N = M + 1$ players: $N - 1 = M$ players with degree 1, and 1 player with degree M. So, $A(\mathbf{d}) = \frac{1}{M+1}[M + M] = \frac{2M}{M+1}$, $H(\mathbf{d}) = \frac{M+1}{M+\frac{1}{M}} = \frac{M(M+1)}{M^2+1}$, and $a(\mathbf{d}) = \left(\frac{2M}{M+1}\right)\left(\frac{M^2+1}{M(M+1)}\right) = \frac{2(M^2+1)}{(M+1)^2} > 1$.*

Note that for a star, $a(\cdot)$ is strictly increasing in M. This is intuitive since, as M increases, the degree of the central player increases, while the degree of each periphery player remains constant (and equal to 1). So, a star network with a greater number of contests/periphery players has a greater degree of asymmetry between the central player and each of the periphery players.

4.1 Equilibrium Behavior

We now characterize "symmetric" equilibrium behavior for quasiregular networks. By symmetric, we mean that players with the same degree follow the same strategy. Let $\overline{g}(\gamma)$ denote the number of players in each contest with degree greater than or equal to γ: $\overline{g}(\gamma) = \sum_{t \geq \gamma} K(t)$. Similarly, let $\overline{N}(\gamma)$ denote the total number of players with degree greater than or equal to γ: $\overline{N}(\gamma) = \sum_{i \in \mathcal{N}} g_{im} \mathbb{1}(d_i \geq \gamma)$.

Proposition 1. *Suppose the network is quasiregular with N players, M contests, and prize degree, g. Then there exists a unique symmetric equilibrium. The n^* active players (the players who exert strictly positive effort) are the players with the highest degrees. Specifically, if the players are ordered so that $d_N \leq d_{N-1} \leq \cdots \leq d_1$, then n^* is the smallest integer such that,*

$$d_{n^*+1} \leq \frac{(\overline{g}(d_{n^*}) - 1)M}{\overline{N}(d_{n^*})},$$

or $n^* = N$ if no such integer exists. The equilibrium total effort in each contest is equal across contests, and is given by:

$$S^* = \frac{(\overline{g}(d_{n^*}) - 1)M}{\overline{N}(d_{n^*})}.$$

Let $\mathbf{d}^* = (d_1, \dots d_{n^*})$ denote the vector of active player degrees. Each active player, $i \leq n^*$, chooses effort,

$$x_i^* = \frac{(\overline{g}(d_{n^*}) - 1)M}{\overline{N}(d_{n^*})} \left[1 - \left(\frac{\overline{g}(d_{n^*}) - 1}{\overline{g}(d_{n^*})} \right) \frac{A(\mathbf{d}^*)}{d_i} \right].$$

Aggregate effort on the network is given by,

$$T^* = (\overline{g}(d_{n^*}) - 1)M \left(1 - \frac{\overline{g}(d_{n^*}) - 1}{\overline{g}(d_{n^*})} a(\mathbf{d}^*)\right).$$

Examples

Here, we apply Proposition 1 to describe equilibrium behavior for each of the examples of quasiregular networks described in Section 3.1.

Biregular Networks

For a biregular network summarized by $[N, d; M, g]$, it holds $A(\mathbf{d}) = H(\mathbf{d}) = d$, and hence, $a(\mathbf{d}) = 1$. Proposition 1 then implies the following equilibrium behavior:

Corollary 1. *Suppose that the network is biregular and summarized by* $[N, d; M, g]$*. In the unique symmetric equilibrium, all players are active and each player chooses effort*

$$x^* = \frac{g-1}{g^2} d.$$

Total effort in each contest is,

$$S^* = \frac{(g-1)M}{N}.$$

Aggregate effort in the network is,

$$T^* = \frac{g-1}{g} M.$$

To illustrate Corollary 1, consider a circle network with $N = M$ players and M contests. In a circle, each player and each prize has degree 2: $g = d = 2$. Individual effort, for any M, is $x^* = \frac{1}{2}$, while aggregate effort on the network is $T^* = \frac{M}{2}$. Next, consider a complete network with N players and M contests. In a complete network, each player has degree M: $d = M$, while each contest has degree N: $g = N$. Thus, individual effort is $x^* = \frac{N-1}{N^2} M$ and aggregate effort on the network is $T^* = \frac{N-1}{N} M$.

Star Networks

For a star network with M contests, it holds $A(\mathbf{d}) = \frac{2M}{M+1}$, and $H(\mathbf{d}) = \frac{M(M+1)}{M^2+1}$. Thus, $a(\mathbf{d}) = \frac{2(M^2+1)}{(M+1)^2}$. Using Proposition 1, equilibrium behavior is described as follows:

Corollary 2. *Suppose the network is a star with M contests. In the unique symmetric equilibrium, all players are active. The effort of each periphery player is,*

$$x^* = \frac{M}{(M+1)^2}.$$

The effort of the central player is,

$$x_C^* = \frac{M^2}{(M+1)^2}.$$

Total effort in each contest is,

$$S^* = \frac{M}{M+1}.$$

Aggregate effort on the network is,

$$T^* = \frac{2M^2}{(M+1)^2}.$$

M-Hybrid Networks

For an M-hybrid network with an underlying network summarized by $[N-1, d; M, g-1]$ it holds: $A(\mathbf{d}) = \frac{1}{N}\left((N-1)d + M\right)$, $H(\mathbf{d}) = \frac{N}{\frac{N-1}{d} + \frac{1}{M}}$. It can be shown that, $a(\mathbf{d}) = \left(\frac{g}{g-1}\right)\left(\frac{(N-1)^2+g-1}{N^2}\right)$.

Corollary 3. *Suppose the network is an M-hybrid with an underlying biregular network summarized by $[N-1, d; M, g-1]$. In the unique symmetric equilibrium, all players are active. The effort of each underlying player is,*

$$x^* = \frac{N-1}{N^2}d.$$

The effort of the hybrid player is,

$$x_H^* = \frac{(N-g+1)(N-1)}{N^2}d.$$

Total effort in each contest is,

$$S^* = \frac{(N-1)}{N}d.$$

Aggregate equilibrium effort in the network is,

$$T^* = \frac{(2N-g)(N-1)}{N^2}d.$$

4.2 Network Structures

We now explore how the network structure affects equilibrium behavior. Our first result shows that equilibrium total effort in each contest is increasing in the prize degree, g.

Proposition 2. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two quasiregular networks, each with N players and M prizes. For network $\mathcal{G}_k$ let g_k and S_k^* denote, respectively, the degree of each contest, and the equilibrium total effort in each contest. If $g_1 < g_2$ then, $S_1^* \leq S_2^*$.*

Proposition 2 establishes that the total effort in each contest is increasing in the degree of each contest. Note, however, that this does not imply that aggregate effort in the network increases. Indeed, it may happen that aggregate effort in the network decreases, even though the total effort in each contest increases. Example 3 illustrates; the two networks described in the example are illustrated in Figure 6.

Example 3. *Fix $N = 6$, $M = 2$, and consider the two network structures depicted in Figure 6. Let $\mathcal{G}_1$ denote the biregular network illustrated in the left panel of the figure. In this network, players 1, 2, and 3 compete in contest 1, while players 4, 5, and 6 compete in contest 2. Thus, $g_1 = 3$. In equilibrium,* *all players are active, and the total effort in each contest is* $S_1^* = \frac{2}{3}$. *Aggregate effort in the network is* $T_1^* = \frac{4}{3}$.

Let $\mathcal{G}_2$ *denote the network illustrated in the right panel of Figure 6. In this network, players 1 and 2 compete only in contest 1, players 5 and 6 compete only in contest 2, while players 3 and 4 compete in both contests. Thus,* $g_2 = 4$; *however, only players 3 and 4 are active in equilibrium. The total effort in each contest is* $S_2^* = 1 > S_1^*$. *The aggregate effort in the network is equal to the total effort in each contest:* $T_2^* = 1 < \frac{4}{3} = T_1^*$.

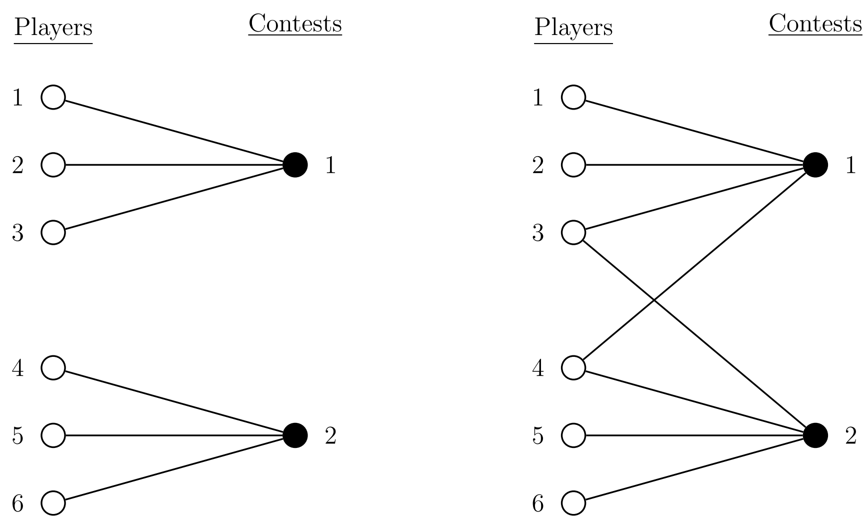

Figure 6: The two network structures described in Example 3.

Adding additional links to a network has two competing effects on aggregate network effort. For those players whose degree increases, effort tends to increase since these players now compete for a greater number of prizes. However, additional links may give rise to a greater degree of asymmetry between players, which reduces the competitiveness in the network. As can be seen in the expression for aggregate network effort given in Proposition 1, as the level of asymmetry rises (as measured by $a(\mathbf{d})$), aggregate network effort tends to decrease, ceteris paribus. The example above illustrates a scenario in which adding additional links leads the network to be asymmetric enough that aggregate network effort decreases.

Of course, additional links may also reduce the level of asymmetry, giving rise to a more competitive contest network. For M-Hybrid networks, for example, an increase in the degree of each contest (resulting from an increase in the degree of each underlying player) reduces the disparity between each of these players and the hybrid player.[9] Our next result establishes that aggregate network effort for M-hybrids is increasing in the degree of each contest.

Proposition 3. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two M-Hybrid networks, each with N players and M prizes. If $g_1 < g_2$ then $T_1^* < T_2^*$.*

Our next result complements Proposition 3, and provides a sufficient condition under which aggregate network effort increases when additional links are added to arbitrary quasiregular networks.

Proposition 4. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two quasiregular networks with N players, and M contests. Let $\mathbf{d}_k$ denote the vector of player degrees in network k. Suppose that, in equilibrium, all players are active in both networks. If $g_1 < g_2$, $a(\mathbf{d}_2) \leq a(\mathbf{d}_1)$ and $a(\mathbf{d}_2) < \frac{g_1 g_2}{g_1 g_2 - 1}$ then, $T_1^* < T_2^*$.*

Intuitively, if additional links are added to a network in such a way that the resulting structure is "symmetric enough" then aggregate equilibrium effort in the network increases. The next example illustrates Proposition 4; Figure 7 provides an illustration of the networks described in the example.

Example 4. *Consider the two networks depicted in Figure 7. In the left panel, we have a star with $M = 5$ contests. In this network it holds, $g = 2$, and $a(\mathbf{d}) \approx 1.444$. The network depicted in the right panel is an augmented star, where each periphery player now competes in two contests. Note that this network is an M-hybrid in which the underlying network is a 5-player circle. In this M-hybrid, it holds, $g = 3$, and $a(\mathbf{d}) = 1.125$. It is straightforward to check that the hypotheses of Proposition 4 are satisfied in this example, with $\mathcal{G}_1$ denoting the star, and $\mathcal{G}_2$ denoting the M-hybrid. Indeed it holds, $T_2 = 2.5 > T_1 \approx 1.389$.*

[9] Indeed, it is straightforward to show that a is strictly decreasing in g for M-hybrid networks.

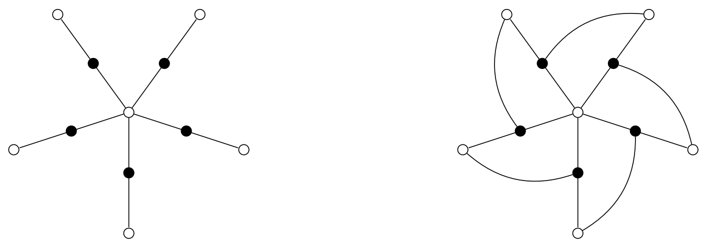

Figure 7: The two network structures described in Example 4.

Proposition 4 is useful for addressing questions of optimal network design. Note that since $a(\cdot) \geq 1$, and for any biregular network, $a(\mathbf{d}) = 1$, the next corollary follows almost immediately from Proposition 4.

Corollary 4. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two quasiregular networks with N players, and M contests. If $\mathcal{G}_2$ is biregular and $g_1 < g_2$ then $T_1^* < T_2^*$.*

Corollary 4 shows that for a fixed number of players and contests, a biregular network induces greater aggregate network effort than any quasiregular network with smaller contest degree. Now since $g \leq N$, and a complete network is biregular with $g = N$, the following result follows immediately by Corollary 4.

Corollary 5. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two quasiregular networks with N players and M contests. If $\mathcal{G}_2$ is the complete network and $g_1 < N$ then $T_1^* < T_2^*$.*

Corollary 5 shows that the complete network induces strictly greater aggregate network effort than any other network.

4.3 Addition/Deletion of a Player

In this section we explore the impact of adding/deleting a player from the network. We begin with an example, which shows that adding a player to the network may have no impact on players' efforts.

Example 5. *Let $\mathcal{G}$ be a three-player circle, and let $\tilde{\mathcal{G}}$ be the network formed by adding an additional player with degree 1. $\tilde{\mathcal{G}}$ is illustrated in Figure 8.*

It may be verified that there is only one symmetric equilibrium, and in this equilibrium, each player with degree 2 exerts effort of $\frac{1}{2}$, while the player with degree 1 is inactive. The effort of each active player when the network is $\tilde{\mathcal{G}}$ is equal to the effort of each player when the network is $\mathcal{G}$.

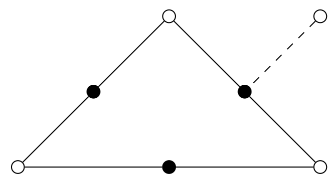

Figure 8: The network, $\tilde{\mathcal{G}}$, described in Example 5. Players are represented by hollow nodes; contests are represented by solid nodes. The dashed line represents the link formed by the new entrant.

Example 5 illustrates the fact that adding a player to the network may have no impact on the efforts of the incumbent players if the additional player's degree is small enough. Our next result generalizes the finding of Example 5, and is an immediate consequence of Theorem 1 and Proposition 1.

Corollary 6. *Let $\mathcal{G}$ be a quasiregular network with N players, M contests, and contest degree, g. Let x_i^* and d^* denote, respectively, the symmetric equilibrium effort of player i and the smallest degree of all active players, when the network is $\mathcal{G}$. Let $\overline{g}(d^*)$ denote the number of players in each contest with degree greater than or equal to d^*. Let $\tilde{\mathcal{G}}$ be the network formed by adding a player, $N+1$, with degree d_{N+1}, and let $\tilde{x}_i^*$ denote the equilibrium effort of player i when the network is $\tilde{\mathcal{G}}$. If*

$$d_{N+1} \leq \frac{(\overline{g}(d^*)-1)M}{\overline{N}(d^*)}$$

then there exists an equilibrium when the network is $\tilde{\mathcal{G}}$ in which $\tilde{x}_{N+1}^* = 0$, and $\tilde{x}_i^* = x_i^*$ for all $i \in \{1, \dots, N\}$.

Example 5 and Corollary 6 reveal that when a player is added to a quasiregular network, the additional player must have a high enough degree in order to have an incentive to actively participate. Our next result examines a scenario in which a player with degree, M, is added to a biregular network. Note that the resulting structure will be an M-hybrid.

Proposition 5. *Let $\mathcal{G}_H$ be an M-hybrid network with an underlying biregular network, $\mathcal{G}_B$, summarized by $[N-1, d; M, g-1]$. Let $\mathbf{d}_H$ denote vector of player degrees in the M-hybrid network, S_H^* (S_B^*) denote equilibrium total effort in each contest, and T_H^* (T_B^*) denote the equilibrium aggregate effort in the network. Then $S_B^* < S_H^*$. Moreover, $T_H^* < T_B^*$ if and only if*

$$1 + \frac{1}{(g-1)^3} < a(\mathbf{d}_H). \tag{6}$$

Proposition 5 demonstrates that the entry of an additional player may result in lower aggregate effort in the network. Condition (6) can be interpreted as saying that the new entrant induces a significant amount of asymmetry in the network. Our result closely relates to the Exclusion Principle (Baye et al., 1993). As discussed in the introduction, it is well-known that the Exclusion Principle does not apply under the Tullock CSF (see, e.g., Fang, 2002; Matros, 2006). However, previous work in the contest literature focuses on one particular pattern of interactions (one in which all players compete for a single prize); the addition/deletion of one player does not affect this structure.[10] In contrast, the impact of adding or deleting a player in our network setting, not only depends on that player's direct contribution, but also depends on indirect effects that stem from the player's influence on the structure of interactions (a similar effect is described in Ballester et al., 2006). When indirect network effects are taken into account, our Proposition 5 suggests that the Exclusion Principle is a more robust phenomenon than previously thought. The following example illustrates our finding. The networks described in the example are illustrated in figure 9.

[10]Two exceptions are Dahm and Esteve-Gonzalez (2014) and Dahm (2017). Both studies explore a particular network structure in which all players compete for a main prize, while a subset of disadvantaged players also compete for a secondary prize. Dahm (2017) shows that excluding an advantaged player altogether may increase total effort under the APA CSF, but even greater effort can be generated by only excluding the advantaged player from competing for the secondary prize.

Example 6. *Suppose $\mathcal{G}_H$ is an M-hybrid with an underlying biregular network, $\mathcal{G}_B$, summarized by $[N-1, d; M, g-1] = [6, 2; 4, 3]$. This network is illustrated in figure 9. It holds that $a(\mathbf{d}_H) = \frac{28}{13} \approx 2.15 > 1 + \frac{1}{(g-1)^3} = 1 + \frac{1}{8^3} = 1.125$. Equilibrium aggregate efforts in each network are $T_H^* = \frac{120}{49} \approx 2.45 < T_B^* = \frac{8}{3} \approx 2.67$.*

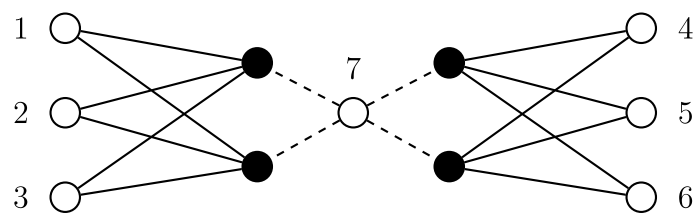

Figure 9: The M-hybrid network described in Example 6. Players are represented by hollow nodes; contests are represented by solid nodes. Player 7 is the hybrid player; removing this player results in a biregular network.

5 Connections to Literature

In this section, we unify our previous results with the existing literature on contests. In particular, we derive equivalences between individual behavior in our network setting, and behavior in single-prize contests.

Asymmetric Contests

To begin, we establish a connection between behavior in quasiregular networks and behavior in single-prize contests, as explored in Stein (2002).

Proposition 6. *Let $\mathcal{G}$ be a quasiregular network with N players, M contests, and prize degree, g. Consider the g players who compete in any particular contest, and re-label these players to be $\{1, \ldots, g\}$. In a symmetric equilibrium in the network game, individual behavior of each player, $i \in \{1, \ldots, g\}$, is equivalent to the behavior in a g-player single-prize asymmetric contest in* *which the prize valuations are* $(v_1, \ldots, v_g) = (d_1, \ldots, d_g)$*. Moreover, equilibrium total effort in each contest of the network game is equal to the total effort in the single-prize contest.*

Proposition 6 draws a connection between behavior in the network game, and behavior in asymmetric, single-prize contests. Each player in the network game behaves "as-if" competing in a g-player single-prize contest in which the players' values for the prize are equal to their degrees in the network. Intuitively, quasiregularity implies that each contest is symmetric, in the sense that each contains the same number of players of any degree, $\gamma \in \{1, \cdots, M\}$. Thus, the degree distribution of the competitors that some player, i, faces is the same across contests. Then, suppose that i's competitors with the same degree follow the same strategy, and note that i's payoff is equal to the sum of her payoffs across individual contests. It follows that for any effort choice, x_i, i's payoff, is equal to what her payoff would be in a single, g-player contest, in which i has a prize value equal to her degree, d_i.

For example, in a star network, each contest contains 1 periphery player with degree 1 and 1 central player with degree M. From the perspective of the central player, each of the M contests is identical, and includes a single competitor with degree 1. When each periphery player follows the same strategy, the payoff to the central player is equal to what her payoff would be in a single 2-player contest in which she has prize value, M.

Symmetric Contests

When all players have the same degree (i.e., when the network is biregular), then as a special case of Proposition 6, it is clear that behavior in the network game is equivalent to the behavior in a single-prize contest, in which all players have the same prize value. The next corollary formalizes this observation.

Corollary 7. *Suppose the network is biregular. Individual equilibrium effort and equilibrium expected payoffs are equivalent to the behavior and expected payoffs in a* g*-player contest in which players compete for a single prize with common value* d*. Aggregate equilibrium effort in the network is equivalent to* *total equilibrium effort in a g-player contest in which players compete for a single prize with common value M.*

Corollary 7 establishes a connection between behavior in the contest played on the network, and behavior in symmetric single-prize contests without network effects. Specifically, if $\mathcal{G}$ is biregular, and summarized by $[N, d; M, g]$, then each player in the network behaves as-if competing in a g-player contest in which each player has common value d, while aggregate equilibrium effort in the network is as-if g players each compete for a prize of common value M.

Discussion

Our findings reveal that a number of results from the contest literature can be obtained by varying the structure of the network in our framework. Table 1 summarizes these linkages.

Table 1: A Unified Framework

Contest Results and Corresponding Network Structures		
Result	Related Literature	Network Structure
Behavior in Asymmetric Contests	Hillman and Riley (1989), Nti (1999, 2004), Stein (2002), Matros (2006)	Quasiregular Networks
Behavior in Symmetric Contests	Tullock (1980), Hillman and Riley (1989)	Biregular Networks
Exclusion Principle	Baye et al. (1993), Fang (2002), Matros (2006), Menicucci (2006)	M-Hybrid Networks

One might also wonder if, for a given single-prize contest, there exists a corresponding (non-trivial) quasiregular network, which induces the same equilibrium behavior. Indeed, it is straightforward to construct such a network. To see this, consider an n-player, single prize contest with prize values $V = (v_1, \ldots, v_n)$, where $v_i \in \mathbb{N}$ for each i; order the players so that $v_n \leq \cdots \leq v_1$. Let ℓ_v be the number of players in the single-prize contest with prize value v.

Now we construct a quasiregular network that will induce the same individual behavior in equilibrium. Choose M – the number of contests – to be the smallest integer such that, for all $v \in \{v_1, \ldots, v_n\}$, $\frac{M\ell_v}{v}$ is an integer. Choose N – the total number of players in the network – such that $N = \sum_{v \in \{v_1,\ldots,v_n\}} \frac{M\ell_v}{v}$. Note that, by construction, $N \in \mathbb{N}$. Choose $K(\gamma)$ – the number of players in each contest with degree γ – such that $K(\gamma) = \ell_\gamma$. To see that our choice of $K(\gamma)$ is consistent with our choices of N and M, note that if there are ℓ_γ players in each of the M contests with degree γ, then there must be a total of $\frac{M\ell_\gamma}{\gamma}$ players in the network with degree, γ. Thus, the total number of players in the network must be $\sum_{\gamma=1}^{M} \frac{M\ell_\gamma}{\gamma} = \sum_{v \in \{v_1,\ldots,v_n\}} \frac{M\ell_v}{v} = N$.

Note that each contest in the quasiregular network described above has $g = n$ players. Choose any one of the contests, and re-label the participants in this contest to be $1, \ldots, n$, ordered so that $d_n \leq \cdots \leq d_1$. By construction $(d_1, \ldots, d_n) = (v_1, \ldots v_n)$. By Proposition 6, individual behavior in each contest of the network game is equivalent to the behavior in the single-prize contest. The following example illustrates:

Example 7. *Consider a 4-player, single prize contest where the prize values are* $(v_1, v_2, v_3, v_4) = (3, 2, 2, 1)$*. Note that* $\ell_1 = \ell_3 = 1$ *and* $\ell_2 = 2$*. The smallest integer,* M*, such that, for all* $v \in \{3, 2, 2, 1\}$*,* $\frac{M\ell_\gamma}{\gamma} \in \mathbb{N}$ *is* $M = 3$*. Then, choose* $N = \sum_{v \in \{3,2,2,1\}} \frac{3\ell_v}{v} = 1 + 3 + 3 = 7$*. Finally, choose* $K(1) = \ell_1 = 1$*,* $K(2) = \ell_2 = 2$*,* $K(3) = \ell_3 = 1$*, and* $K(\gamma) = \ell_\gamma = 0$*, for* $\gamma > 3$*. The resulting quasiregular network is depicted in Figure 2. Moreover, each player in the network game behaves as-if competing in the single-prize contest.*

6 Conclusion

In this paper we proposed a framework for studying contests on networks. We characterized equilibrium in terms of the underlying network structure, and studied how this structure affects equilibrium behavior. Furthermore, we have shown that a number of results from the contest literature may be obtained in our framework by varying the structure of the network. In addition, we have provided a new exclusion result, akin to Baye et al.'s (1993) Exclusion

Principle, but which is relevant under the Tullock CSF. This result contrasts the existing literature, and highlights the relevance of network effects in our model.

7 Appendix

Proof of Proposition 1

Let $(x_1^*, \cdots, x_N^*)$ denote a symmetric equilibrium effort profile, and let $x(\gamma)$ denote the effort choice of all players with degree γ. Let S_m denote the total effort in contest m. For any m it holds:

$$S_m = \sum_{i \in \mathcal{N}} x_i g_{im} = \sum_{\gamma=1}^{M} x(\gamma) K(\gamma).$$

Hence, total effort in each contest is the same: $S_1 = \cdots = S_M = S$ for some S. It should be clear that, in equilibrium, S must be strictly positive: $S > 0$.

Let $\mathcal{N}^* \subseteq \mathcal{N}$ denote the set of active players, and let $n^* = |\mathcal{N}^*|$. Choose some player $i \in \mathcal{N}^*$. Note that by the strict concavity of i's payoff function, the first order condition is necessary and sufficient to characterize player i's optimal effort choice, for a given vector of efforts from the other players. Hence, x_i^* must satisfy the following first-order condition:

$$\frac{S - x_i^*}{S^2} g_{i1} + \cdots + \frac{S - x_i^*}{S^2} g_{iM} = 1,$$

or

$$\frac{S - x_i^*}{S^2} (g_{i1} + \cdots + g_{iM}) = \frac{S - x_i^*}{S^2} d_i = 1,$$

and hence,

$$x_i^* = S - S^2 \frac{1}{d_i}.$$

Note that player i is active if and only if $d_i > S$. Now see that the number of active players in contest m is, $\sum_{i \in \mathcal{N}^*} g_{im} = \sum_{\gamma=1}^{M} K(\gamma)\mathbb{1}(x(\gamma) > 0)$. Thus, the number of active players in each contest must be the same across contests; call this number g^*. It follows that,

$$S = \sum_{i \in \mathcal{N}^*} g_{im} x_i^* = g^* S - S^2 \sum_{i \in \mathcal{N}^*} \frac{g_{im}}{d_i}. \tag{7}$$

Now note that for any contest, m, it holds,

$$\sum_{i \in \mathcal{N}^*} \frac{g_{im}}{d_i} = \sum_{\gamma=1}^{M} \frac{K(\gamma)}{\gamma} \mathbb{1}(x(\gamma) > 0).$$

Hence, $\sum_{i \in \mathcal{N}^*} \frac{g_{im}}{d_i}$ is equal across contests; let β denote this sum. It holds,

$$M\beta = \sum_{m \in \mathcal{M}} \sum_{i \in \mathcal{N}^*} \frac{g_{im}}{d_i} = \sum_{i \in \mathcal{N}^*} \frac{1}{d_i} \sum_{m \in \mathcal{M}} g_{im} = \sum_{i \in \mathcal{N}^*} \frac{1}{d_i} d_i = n^*.$$

Hence, $\beta = \frac{n^*}{M}$. Equation (7) then yields,

$$S = g^* S - S^2 \frac{n^*}{M}.$$

Solving for S,

$$S = \frac{(g^* - 1)M}{n^*}.$$

The arguments above reveal that, for a given set of active players, the symmetric equilibrium individual efforts are uniquely determined. We now show that the set of active players is unique. We proceed by contradiction. So, suppose there are two distinct sets of active players, $\mathcal{N}_1^*$ and $\mathcal{N}_2^*$, that may arise in equilibrium. Since our contradiction hypothesis is that $\mathcal{N}_1^* \neq \mathcal{N}_2^*$, there must be some player $\hat{\imath}$ that is active in one equilibrium, but inactive in the other. Without loss of generality, suppose $\hat{\imath} \in \mathcal{N}_1^*$ but $\hat{\imath} \notin \mathcal{N}_2^*$. Let $n_k^* = |\mathcal{N}_k^*|$, and let g_k^* denote the number of active players in each contest when the set of active players is $\mathcal{N}_k^*$. Finally, let $S_k = \frac{(g_k^* - 1)M}{n_k^*}$ denote the equilibrium total effort in each contest.

Since $\hat{\imath}$ is inactive (active) when the set of active players is $\mathcal{N}_2^*$ ($\mathcal{N}_1^*$), it must be that $S_2 \geq d_{\hat{i}} > S_1$. Moreover, note that for any $j < \hat{i}$, $d_j \geq d_i > S_1$, and hence $j \in \mathcal{N}_1^*$. It follows that $n_1^* \geq \hat{i}$, and $\{1, \ldots, \hat{i}\} \subseteq \mathcal{N}_1^*$. Similarly, for any $j > \hat{i}$, $d_j \leq d_{\hat{i}} < S_2$, and hence, $j \notin \mathcal{N}_2^*$. It follows that $n_2^* \leq \hat{i} - 1$ and $\mathcal{N}_2^* \subseteq \{1, \ldots, \hat{i} - 1\}$. Let $\Delta = n_1^* - n_2^* \geq 1$.

Note that for $k = 1, 2$ it must hold that, $\sum_{i \in \mathcal{N}_k^*} d_i = M g_k^*$. The RHS is the total number of links from prizes to active players. The LHS is the total number of links from active players to prizes. Thus,

$$
\begin{aligned}
g_1^* &= \frac{1}{M} \sum_{i=1}^{n_2^*} d_i + \frac{1}{M} \sum_{i=n_2^*}^{n_1^*} d_i \\
&\geq g_2^* + \frac{1}{M} \sum_{i=n_2^*}^{n_1^*} d_{n_1^*} = g_2^* + \Delta \frac{d_{n_1^*}}{M}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
S_1 - S_2 &= \frac{(g_1^* - 1)M}{n_1^*} - \frac{(g_2^* - 1)M}{n_2^*} \\
&\geq \frac{M}{n_1^* n_2^*} \left(\Delta + n_2^* g_2^* + n_2^* \Delta \frac{d_{n_1^*}}{M} - n_1^* g_2 \right) \\
&= \frac{\Delta}{n_1^*} \left(d_{n_1^*} - \frac{M(g_2^* - 1)}{n_2^*} \right) \\
&> \frac{\Delta}{n_1^*} (S_1 - S_2) > S_1 - S_2.
\end{aligned}
$$

The first inequality follows since $g_1^* \geq g_2^* + \Delta \frac{d_{n_1^*}}{M}$. The first strict inequality holds since $\Delta > 0$, $S_2 = \frac{M(g_2^* - 1)}{n_2^*}$, and, by definition, player n_1^* is active when the set of active players is $\mathcal{N}_1^*$, which means $d_{n_1^*} > S_1$. The final inequality holds since $S_1 < S_2$, and $\frac{\Delta}{n_1^*} = \frac{n_1^* - n_2^*}{n_1^*} < 1$. The hypothesis that $\mathcal{N}_1^* \neq \mathcal{N}_2^*$ has led to a contradiction. Therefore, the symmetric equilibrium set of active players must be unique. In particular, $\mathcal{N}^*$ must consist of the players, $\{1, \ldots, n^*\}$ with the n^* highest degrees, where n^* is the smallest integer such that,

$$d_{n^*+1} \leq \frac{(\overline{g}(d_{n^*})-1)M}{\overline{N}(d_{n^*})}$$

or $n^* = N$ if no such integer exists. We have thus constructed a candidate equilibrium profile, $(x_1^*, \cdots, x_N^*)$, where:

$$x_i^* = \begin{cases} S - S^2 \frac{1}{d_i} & i \leq n^* \\ 0 & i > n^* \end{cases}$$

and

$$S = \frac{(\overline{g}(d_{n^*})-1)M}{\overline{N}(d_{n^*})}.$$

Assuming all players $j \neq i$ follow this strategy, it is straightforward to show that player i maximizes her payoff by following x_i^*. Thus, the profile above indeed constitutes a symmetric equilibrium. Since $\mathcal{N}^*$ is uniquely determined when players follow symmetric strategies, and for a given set of active players S and x_i^* are uniquely determined, it follows that this symmetric equilibrium is unique. Finally, by summing x_i^* over i it is clear that the expression for total effort is as given in the proposition. □

Proof of Proposition 2

As will be shown in the proof of Proposition 6, S_k^* is equal to the equilibrium total effort in a g_k-player single-prize asymmetric contest. As shown by Matros (2006), the total effort in this single prize contest is increasing in the number of players. Thus, $g_2 > g_1$ implies $S_2^* \geq S_1^*$. □

Proof of Proposition 3

By Corollary 3, for network $\mathcal{G}_k$, $T_k^* = \frac{(2N-g_k)(N-1)}{N^2} d_k$, where d_k is the degree of each underlying player. By the link property, (4), it holds $N(d_k - 1) = M(g_k - 1)$. Thus, $T_k^* = \frac{(2N-g_k)(g_k-1)}{N^2} M$. Note that for $g < N$, the expression $(2N-g)(g-1)$ is strictly increasing in g. As $g_1 < g_2 \leq N$, it is straightforward to see that $T_1^* < T_2^*$. □

Proof of Proposition 4

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two quasiregular networks, and suppose all players are active for both networks. Suppose $g_1 < g_2$, $a_2 \leq a_1$, and $a_2 < \frac{g_1 g_2}{g_1 g_2 - 1}$, where g_k denotes the degree of each contest in network k, and a_k denotes $a(\mathbf{d}_k)$. By Proposition 1, aggregate effort when the network is $\mathcal{G}_k$ is given by, $T_k^* = M\left[g_k - 1 - \frac{(g_k-1)^2}{g_k} a_k\right]$. It follows,

$$\begin{aligned}\frac{1}{M}(T_2^* - T_1^*) &= g_2 - g_1 - \frac{(g_2-1)^2}{g_2} a_2 + \frac{(g_1-1)^2}{g_1} a_1 \\ &\geq g_2 - g_1 - a_2 \left[\frac{(g_2-1)^2}{g_2} - \frac{(g_1-1)^2}{g_1}\right] \\ &= g_2 - g_1 - a_2 \left(\frac{g_2 - g_1}{g_1 g_2}\right)(g_1 g_2 - 1) > 0.\end{aligned}$$

The first inequality follows since $a_2 \leq a_1$. The final inequality holds since $g_2 > g_1$ and $a_2 < \frac{g_1 g_2}{g_1 g_2 - 1}$. □

Proof of Proposition 5

By Proposition 1, equilibrium aggregate effort on the M-Hybrid network is $T_H^* = M(g-1)\left[1 - \frac{g-1}{g} a(\mathbf{d}_H)\right]$. By Corollary 1, equilibrium aggregate effort on the underlying biregular network is $T_B^* = \frac{g-2}{g-1} M$. Then, $T_B^* > T_H^*$ if and only if,

$$\begin{aligned}\frac{g-2}{g-1} M > M(g-1)\left[1 - \frac{g-1}{g} a(\mathbf{d}_H)\right] \iff \\ a(\mathbf{d}_H) > 1 + \frac{1}{(g-1)^3}.\end{aligned}$$

□

Proof of Proposition 6

Let $\mathcal{G}$ be the network described in the proposition. Let let $\mathcal{N}^*$ denote the equilibrium set of active players. Further, let $n^* = |\mathcal{N}^*|$ and let g^* be the number of active players in each contest. By Proposition 1, it holds that for $i \leq n^*$, $x_i = \frac{(g^*-1)M}{n^*}$. Let $h_m(\gamma)$ denote the harmonic mean of the degrees of players with degree greater than or equal to γ who compete in contest m. Note that quasiregularity implies that this number is the same across contests: For each m, $h_m(\gamma) = h(\gamma)$. In particular, $h(\gamma)$ is given by,

$$h(\gamma) = \frac{\overline{g}(\gamma)}{\sum_{i \in \mathcal{N}} \mathbb{1}(d_i \geq \gamma) \frac{g_{im}}{d_i}} = \frac{\overline{g}(\gamma)}{\sum_{t \geq \gamma} \frac{K(t)}{t}}.$$

In equilibrium, the harmonic mean of the active player degrees in each contest is given by,

$$h(d_{n^*}) = \frac{\overline{g}(d_{n^*})}{\sum_{i \in \mathcal{N}} \mathbb{1}(d_i \geq d_{n^*}) \frac{g_{im}}{d_i}} = \frac{g^*}{\sum_{i \in \mathcal{N}^*} \frac{g_{im}}{d_i}}.$$

As shown in the proof of Proposition 1, $\sum_{i \in \mathcal{N}^*} \frac{g_{im}}{d_i} = \frac{n^*}{M}$. Hence, $h(d_{n^*}) = \frac{Mg^*}{n^*}$. Note that if we take any one particular contest, and relabel the players in this contest to be $\{1, \ldots, g\}$ where $d_1 \geq \cdots \geq d_g$, then the players with the g^* highest degrees are those players with degree greater than or equal to d_{n^*} under the original ordering. Then, under the new ordering, the equilibrium total effort in any contest can be written,

$$S = \frac{g^* - 1}{g^*} h(d_{g^*}).$$

Using this expression, and comparing the individual equilibrium effort with the equilibrium characterized by Stein (2002), it is straightforward to show that the equilibrium behavior of player $i \in \{1, \ldots, g\}$ is identical to a single-prize contest in which player valuations are $(d_1, \ldots, d_g)$, where $(d_1, \ldots, d_g)$ is the (re-labled) profile of player degrees in some particular contest in the network game. It follows immediately that the total equilibrium effort in each contest is equal to the total equilibrium effort in the single-prize asymmetric contest. □

References

Ballester, C., Armengol, A. C., and Zenou, Y. (2006). Who's who in networks. Wanted: The key player. *Econometrica*, 74(5):1403–1417.

Baye, M. R. and Hoppe, H. C. (2003). The strategic equivalence of rent-seeking, innovation, and patent-race games. *Games and Economic Behavior*, 44(2):217–226.

Baye, M. R., Kovenock, D., and De Vries, C. G. (1993). Rigging the lobbying process: An application of the all-pay auction. *The American Economic Review*, 83(1):289–294.

Bramoullé, Y. and Kranton, R. (2007). Public goods in networks. *Journal of Economic Theory*, 135(1):478–494.

Bramoullé, Y. and Kranton, R. (2016). Games played on networks. In Bramoullé, Yann, A. G. and Rogers, B., editors, *The Oxford Handbook on the Economics of Networks*. Oxford University Press, Oxford.

Bramoullé, Y., Kranton, R., and D'Amours, M. (2014). Strategic interaction and networks. *The American Economic Review*, 104(3):898–930.

Calvó-Armengol, A. (2004). Job contact networks. *Journal of Economic Theory*, 115(1):191–206.

Calvó-Armengol, A. and Jackson, M. O. (2004). The effects of social networks on employment and inequality. *The American Economic Review*, 94(3):426–454.

Calvó-Armengol, A. and Zenou, Y. (2004). Social networks and crime decisions: The role of social structure in facilitating delinquent behavior. *International Economic Review*, 45(3):939–958.

Che, Y.-K. and Gale, I. (2003). Optimal design of research contests. *The American Economic Review*, 93(3):646–671.

Congleton, R., Hillman, A., and Konrad, K. (2008). *Forty years of rent-seeking research*, volume 1-2. Springer & Verlag, Heidelberg.

Dahm, M. (2017). All-pay auctions with extra prize: A partial exclusion principle. CeDEx Discussion Paper No. 2017-01.

Dahm, M. and Esteve-Gonzalez, P. (2014). Affirmative action through extra prizes. CeDEx Discussion Paper No. 2014-08.

Fang, H. (2002). Lottery versus all-pay auction models of lobbying. *Public Choice*, 112(3-4):351–371.

Franke, J. and Öztürk, T. (2015). Conflict networks. *Journal of Public Economics*, 126:104–113.

Galeotti, A., Goyal, S., Jackson, M. O., Vega-Redondo, F., and Yariv, L. (2010). Network games. *The Review of Economic Studies*, 77(1):218–244.

Gassmann, O. and Von Zedtwitz, M. (1999). New concepts and trends in international R&D organization. *Research Policy*, 28(2-3):231–250.

Goyal, S. and Joshi, S. (2003). Networks of collaboration in oligopoly. *Games and Economic Behavior*, 43(1):57–85.

Goyal, S. and Moraga-Gonzalez, J. L. (2001). R&D networks. *RAND Journal of Economics*, 32(4):686–707.

Grandjean, G., Tellone, D., and Vergrote, W. (2016). Cooperation, competition and entry in a Tullock contest. CORE Discussion Paper 2016/32.

Hillman, A. L. and Riley, J. G. (1989). Politically contestable rents and transfers. *Economics & Politics*, 1(1):17–39.

Huremovic, K. (2016). A noncooperative model of contest network formation. HAL working paper (halshs-01154676v2).

Jackson, M. O. (2008). *Social and Economic Networks*. Princeton University Press, Princeton.

Jackson, M. O. and Nei, S. (2015). Networks of military alliances, wars, and international trade. *Proceedings of the National Academy of Sciences*, 112(50):15277–15284.

Jackson, M. O. and Yariv, L. (2007). Diffusion of behavior and equilibrium properties in network games. *The American Economic Review*, 97(2):92–98.

Jackson, M. O. and Zenou, Y. (2014). Games on networks. In Young, P. and Zamir, S., editors, *Handbook of Game Theory*, volume 4. Elsevier.

König, M. D., Rohner, D., Thoenig, M., and Zilibotti, F. (2017). Networks in conflict: Theory and evidence from the great war of africa. *Econometrica*, 85(4):1093–1132.

Konrad, K. (2009). *Strategy and Dynamics in Contests*. Oxford University Press, New York, NY.

Kovenock, D. and Roberson, B. (2018). The optimal defense of networks of targets. *Economic Inquiry, Forthcoming.*

Marinucci, M. and Vergote, W. (2011). Endogenous network formation in patent contests and its role as a barrier to entry. *The Journal of Industrial Economics*, 59(4):529–551.

Matros, A. (2006). Rent-seeking with asymmetric valuations: Addition or deletion of a player. *Public Choice*, 129(3-4):369–380.

Menicucci, D. (2006). Banning bidders from all-pay auctions. *Economic Theory*, 29(1):89–94.

Nitzan, S. (1994). Modelling rent-seeking contests. *European Journal of Political Economy*, 10(1):41–60.

Nti, K. O. (1999). Rent-seeking with asymmetric valuations. *Public Choice*, 98(3-4):415–430.

Nti, K. O. (2004). Maximum efforts in contests with asymmetric valuations. *European Journal of Political Economy*, 20(4):1059–1066.

Stein, W. E. (2002). Asymmetric Rent-Seeking with More than Two Contestants. *Public Choice*, 113(3-4):325–336.

Tullock, G. (1980). Efficient rent-seeking. In Buchanan, J. M., Tollison, R. D., and Tullock, G., editors, *Toward a theory of the rent-seeking society*, number 4, pages 97–112. Texas A&M University Press, College Station, Texas.

Xu, J. and Zhou, J. (2018). Nexus of conflicts: A variational inequality approach. Working paper `http://dx.doi.org/10.2139/ssrn.3029048`.